A COMPANION

TO ANY ELEMENTARY WORK ON

PLANE TRIGONOMETRY:

BUT MORE ESPECIALLY TO THAT OF

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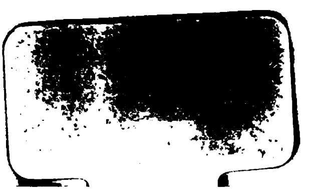
PREFACE.

The Authors have endeavoured to supply a want felt by students in Trigonometry when a list of difficult questions, having only a remote connection with each other, is placed before them for solution.

One question is worked out in full, *as a type*, and others similar to it are given as examples for the student to solve without further assistance.

They cannot hope that their work will be found entirely free from defects, both in execution and design, when subjected to a severe criticism: still they have done their best to be useful and accurate; and they believe that the work will be a great help to many students who do not possess the advantage of a Tutor or Naval Instructor to assist them in the various difficulties that may occur.

They have aimed to make it, what its title implies it to be, a useful companion to any elementary work on



ERRATA.

PAGE.

35. Line 8 from bottom, *for* $\cos.^2$, *read* $\cos.^2 x$.

49. Twice, *for* Art. (5), *read* Art. (2), p. 47.

50. Line 2, *for* $\frac{\sqrt{5-1}}{4}$, *read* $\frac{\sqrt{5}-1}{4}$.

50. Ex. 2, *for* $-$, *read* $+$, *in the middle of the parenthesis.*

51. Line 4, *for* $+ \sin. (30 + B)$, *read* $+ \sin. (30 - B)$.

52 Line 5 from bottom, *for* $\sin. (36 + A) - (36 - A)$, *read* $\sin. (36 + A) - \sin. (36 - A)$.

77. Bottom, *for* p. 70, *read* p. 69.

79. Art. 13, *for* $\triangle$ E F G, *read* Triangle E F G.

The Authors regret to see that the name of Mr. Jeans has been inadvertently mis-spelt in several places. They would be glad to have any errors of the press pointed out to them which may have escaped their notice.

(A.)

$$\text{Sin. A . Cosec. A} = 1 \quad . \quad . \quad . \quad . \quad 1$$

$$\text{Cos. A . Sec. A} = 1 \quad . \quad . \quad . \quad . \quad 2$$

$$\text{Tan. A . Cot. A} = 1 \quad . \quad . \quad . \quad . \quad 3$$

$$\text{Sin.}^2 \text{A} + \text{Cos.}^2 \text{A} = 1 \quad . \quad . \quad . \quad . \quad 4$$

$$\text{Sec.}^2 \text{A} - \text{Tan.}^2 \text{A} = 1 \quad . \quad . \quad . \quad . \quad 5$$

$$\text{Cosec.}^2 \text{A} - \text{Cot.}^2 \text{A} = 1 \quad . \quad . \quad . \quad . \quad 6$$

$$\text{Tan. A} = \frac{\text{sin. A}}{\text{cos. A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad 7$$

$$1 - \text{Cos. A} = \text{Vers. A} \quad . \quad . \quad . \quad 8$$

(1). The following relations are obvious from the above formulæ, simply by dividing—

$$\text{Sin. A} = \frac{1}{\text{cosec. A}}; \ \text{cos. A} = \frac{1}{\text{sec. A}};$$

$$\text{tan. A} = \frac{1}{\text{cot. A}}; \ \text{cot. A} = \frac{\text{cos. A}}{\text{sin. A}}.$$

Sin. $A = \sqrt{1 - \cos.^2 A}$, and cos. $A = \sqrt{1 - \sin.^2 A}$.

Sec. $A = \sqrt{1 + \tan.^2 A}$, and tan. $A = \sqrt{\sec.^2 A - 1}$.

Cosec. $A = \sqrt{1 + \cot.^2 A}$, and cot. $A = \sqrt{\operatorname{cosec.}^2 A - 1}$.

(2). Trigonometrical formulæ can be frequently simplified by reducing them from fractional to integral forms; for this purpose, the formulæ in Art. (1) are of great use.

Reduce $\dfrac{1}{\sin. A . \cos. A\,(1 - \operatorname{vers.} A)}$ to an integral form.

Since cosec. $A = \dfrac{1}{\sin. A}$, and sec. $A = \dfrac{1}{\cos. A}$,

and $1 - \operatorname{vers.} A = 1 - 1 + \cos. A = \cos. A$, we have

$$\frac{1}{\sin. A \cos. A\,(1 - \operatorname{vers.} A)} = \frac{1}{\sin. A}\,\frac{1}{\cos.^2 A} = \operatorname{cosec.} A \sec.^2 A.$$

the integral form required. (See Jeane's Trig., p. 6.)

Transform the following fractions to integral forms :—

1. $\dfrac{1}{\sin.^2 A \cos.^2 A}$. 2. $\dfrac{1}{\sin. A\,(1 - \operatorname{vers.} A)}$.

3. $\dfrac{1}{\cos.^2 A \sqrt{1 - \cos.^2 A}}$. 4. $\dfrac{1}{\sin. A \sqrt{1 - \sin.^2 A}}$.

5. $\frac{\sin. A}{(1 - \text{vers.} A) \sqrt{1 - \cos.^2 A}}$.

6. $\frac{\cos. A}{\sin.^2 A \sqrt{1 - \cos.^2 A} \sqrt{1 - \sin.^2 A}}$.

(3). Reduce $\frac{1}{\tan. A \,.\, \cot.^2 A \sqrt{\text{cosec.}^2 A - 1}}$ to an integral form.

Since $\cot. A = \frac{1}{\tan. A}$, and $\tan. A = \frac{1}{\cot. A}$, and

$$\sqrt{\text{cosec.}^2 A - 1} = \cot. A;$$

$$\therefore \frac{1}{\tan. A \,.\, \cot.^2 A \sqrt{\text{cosec.}^2 A - 1}} = \cot. A \tan^2 A \tan. A = \tan.^2 A.$$

Transform the following fractions into integral forms :—

1. $\frac{\tan. A}{\cot.^3 A, \sec. B \sqrt{1 + \cot.^2 B}}$.

2. $\frac{\sec. A}{\sec.^2 A, \text{cosec.}^2 B \sqrt{1 + \tan.^2 A}}$.

3. $\frac{\text{cosec.} A}{(1 - \text{vers.} A)^2 \sqrt{1 + \cot.^2 B}}$.

4. $\frac{\sin. A, \cos. A}{\sec.^2 A, \text{cosec.}^2 A \sqrt{1 - \sin.^2 A}}$.

5. $\dfrac{\sin.^2 A\,(1 - \text{vers. } B)}{(1 - \text{vers. } B)^2 \sqrt{1 + \cot.^2 B}}$.

6. $\dfrac{\cos.^2 A \sqrt{1 - \sin.^2 A}}{\sec. A \tan. A \cot.^2 A \sqrt{\sec.^2 A - 1}}$.

ANSWERS TO ART. (2).

1. Cosec.^{2} A . sec.^{2} A.
2. Cosec. A . sec. A.
3. Sec.^{2} A . cosec. A.
4. Cosec. A . sec. A.
5. Sec. A.
6. Cosec.^{2} A.

ANSWERS TO ART. (3).

1. Tan.^{4} A, cos. B, sin. B.
2. Cos.^{2} A, sin.^{2} B.
3. Cosec. A, sec.^{2} A, sin. B.
4. Sin.^{2} A, cos.^{2} A.
5. Sin.^{2} A, sec. B, sin. B.
6. Cos.^{4} A.

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(4). If sin. A $= \frac{1}{3}$, find the cos. A, tan. A, sec. A.

$$\text{Cos. } A = \sqrt{1 - \sin.^2 A} = \sqrt{1 - \frac{1}{9}} = \frac{2}{3}\sqrt{2}.$$

$$\text{Tan. } A = \frac{\sin. A}{\cos. A.} = \frac{1}{3} \times \frac{3}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

$$\text{Sec. } A = \sqrt{1 + \tan.^2 A} = \sqrt{1 + \frac{1}{8}} = \frac{3}{2\sqrt{2}}.$$

Given, $\frac{\sin. A}{\sin. x} = \cos. A$, to find cosec. x, sin. x, tan x. (Jeane's Trig., p. 8. Q. 22.)

From the given eq. $\frac{\sin. A}{\cos. A} = \sin. x$;

$$\therefore \sin. x = \tan. A.$$

But, $\operatorname{cosec}. x = \frac{1}{\sin. x} = \frac{1}{\tan. A} = \cot. A$;

$$\therefore \tan. x = \frac{\sin. x}{\cos. x} = \frac{\tan. A}{\sqrt{1 - \sin.^2 x}} = \frac{\tan. A}{\sqrt{1 - \tan.^2 A}}.$$

Given, $\frac{\sin. A \cos. B}{\operatorname{cosec}. x} = \frac{\cos. C \sin. B}{\tan. D}$. (Jeane's Trig. p. 9. Q. 25.) Find sin. x.

$$\text{Since } \sin. x = \frac{1}{\operatorname{cosec}. x};$$

$$\therefore \sin. x = \frac{\cos. C \sin. B}{\tan. D \sin. A \cos. B} = \operatorname{cosec}. A \tan. B \cos. C. \cot. D.$$

(5). Given, $\frac{\sin. A}{\cos. x} = \frac{\cos. A}{\sin. x}$, find tan. x, sin. x, cos. x.

Since, $\frac{\sin. A}{\cos. x} = \frac{\cos. A}{\sin. x}$; $\therefore \frac{\sin. x}{\cos. x} = \frac{\cos. A}{\sin. A}$;

$$\therefore \tan. x = \cot. A.$$

$$\text{But } \cos. x = \frac{1}{\sec. x} = \frac{1}{\sqrt{1 + \tan.^2 x}} = \frac{1}{\sqrt{1 + \cot.^2 A}},$$

$$\text{and } \sin. x = \frac{1}{\operatorname{cosec.} x} = \frac{1}{\sqrt{1 + \cot.^2 x}} = \frac{1}{\sqrt{1 + \tan.^2 A}}.$$

The sec. x, and cosec. x, and cot. x, may always be readily obtained from the Eqs., Art. (1).

Solve the following:

1. Given, $\dfrac{\sec. A}{\operatorname{cosec.} x} = \dfrac{\operatorname{cosec.} A}{\sec. x}$. Find tan. x, sin. x, and cos. x.

2. Given, $\dfrac{\tan. A}{\cot. x} = \dfrac{\cot. A}{\tan. x}$. Find tan. x, sin. x, and cos. x.

3. Given, $\dfrac{\operatorname{cosec.} A}{\sec. x} = \dfrac{\sec. A}{\operatorname{cosec.} x}$. Find tan. x, sin. x, and cos. x.

4. Given, $\dfrac{1}{\sin. x} = 4$. Find cos. x, sin. x, tan. x.

ANSWERS.

1. Cot. A, $\dfrac{1}{\sqrt{1 + \tan.^2 A}}, \dfrac{1}{\sqrt{1 + \cot.^2 A}}$.

2. $\pm$ cot. A, $\dfrac{1}{\sqrt{1 + \tan.^2 A}}, \dfrac{1}{\sqrt{1 + \cot.^2 A}}$.

3. The same as in (1).

4. $\dfrac{\sqrt{15}}{4}, \dfrac{1}{4}, \dfrac{1}{\sqrt{15}}$.

(6). Given, $\frac{\sqrt{1-\cos.^2 A}}{\text{cosec. } x} = \sin. A.$ Find sin. x, cos. x, tan. x. (Jeane's Trig., p. 8. Q. 23.)

Since $\sqrt{1 - \cos.^2 A} = \sin. A$, and $\sin. x = \frac{1}{\text{cosec. } x}$;

$$\therefore \sin. A \sin. x = \sin. A;$$

$$\therefore \sin. x = 1.$$

But $\cos. x = \sqrt{1 - \sin.^2 x} = \sqrt{1 - 1} = 0$,

and $\tan. x = \frac{\sin. x}{\cos. x} = \frac{1}{0} =$ infinity.

The expression (1 divided by zero is infinity) may require a little explanation.

1	divided by	1	gives a quotient	1
1	"	$\frac{1}{2}$	"	2
1	"	$\frac{1}{3}$	"	3
1	"	$\frac{1}{10}$	"	10
1	"	$\frac{1}{100}$	"	100
1	"	$\frac{1}{1000}$	"	1000
		&c.	&c.	&c.

Hence, it is readily seen that the quotients increase as the divisors decrease; and when the

divisor is very small the quotient is very large; and when the divisor is an exceedingly small number, the quotient is an exceedingly large number; and when the divisor is zero in the limit, then the quotient is infinity in the limit.

$$\text{The cosec. } x = \frac{1}{\text{sin. } x} = 1.$$

$$\text{sec. } x = \frac{1}{\text{cos. } x} = \frac{1}{0} = \text{infinity}.$$

$$\text{cot. } x = \frac{1}{\text{tan. } x} = \frac{1}{\text{infinity}.} = 0.$$

Solve the following :—

1. Given, sin. $x = \frac{3}{5}$. Find cos. x, tan. x, sec. x, cosec. x, cot. x.

2. Given, cos. $x = \frac{9}{15}$. Find sin. x, tan. x, sec. x, cosec. x, cot. x.

3. Given, tan. $x = 3$. Find sin. x, cos. x, cot. x, sec. x, cosec. x.

4. Given, cot. $x = 4$. Find sin. x, cos. x, tan. x, sec. x, cosec. x.

5. Given, sec. $x = 3$. Find sin. x, cos. x, cot. x, tan. x, cosec. x.

6. Given, cosec. $x = 2$. Find sin. x, cos. x, tan. x, cot. x, sec. x.

ANSWERS.

1. $\frac{4}{5}, \frac{3}{4}, \frac{5}{4}, \frac{5}{3}, \frac{4}{3}$.

2. $\frac{4}{5}, \frac{4}{3}, \frac{5}{3}, \frac{5}{4}, \frac{3}{4}$.

3. $\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{1}{3}, \sqrt{10}, \frac{\sqrt{10}}{3}$.

4. $\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}, \frac{1}{4}, \frac{\sqrt{17}}{4}, \sqrt{17}$.

5. $\frac{2}{3}\sqrt{2}, \frac{1}{3}, \frac{\sqrt{2}}{4}, 2\sqrt{2}, \frac{3}{4}\sqrt{2}$.

6. $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}, \sqrt{3}, \frac{2\sqrt{3}}{3}$.

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(7). If $\frac{(\sin.^2 A + \cos.^2 A) \tan. C}{1 - \cos. x} = \frac{1}{\cot. C}$. Find sin. x, cos. x, tan. x, cot. x, sec. x, cosec. x. (Jeane's Trig., p. 9. Q. 30).

Since $\sin.^2 A + \cos.^2 A = 1$, and $\tan. C \cot. C = 1$;

$$\therefore 1 - \cos. x = 1; \therefore \cos. x = 0.$$

$$\text{Sin. } x = \sqrt{1 - \cos.^2 x} = 1, \text{ and } \tan. x = \frac{\sin. x}{\cos. x} = \frac{1}{0}$$
$$= \text{infinity.}$$

$$\text{Cot. } x = \frac{1}{\tan. x} = \frac{1}{\text{infinity}} = 0.$$

$$\text{Sec. } x = \frac{1}{\cos. x} = \frac{1}{0} = \text{infinity}.$$

$$\text{Cosec. } x = \frac{1}{\sin. x} = \frac{1}{1} = 1.$$

Since $1 - \cos. x = \text{vers. } x$; $\therefore$ $\text{vers. } x = 1$.

(8). Vers. $x = -\dfrac{\sin.^2 A}{1 - \sin.^2 A}$. Find the other trigonometrical functions of x.

$$\text{Vers. } x = 1 - \cos. x = -\frac{\sin.^2 A}{\cos.^2 A} = -\tan.^2 A;$$

$$\therefore \cos. x = 1 + \tan.^2 A = \sec.^2 A.$$

$$\text{Sin. } x = \sqrt{1 - \cos.^2 x} = \sqrt{1 - \sec.^4 A}.$$

$$\text{Tan. } x = \frac{\sin. x}{\cos. x} = \frac{\sqrt{1 - \sec.^4 A}}{\sec.^2 A} = \sqrt{\frac{1 - \sec.^4 A}{\sec.^4 A}}.$$

$$\text{Cot. } x = \frac{1}{\tan. x} = \sqrt{\frac{\sec.^4 A}{1 - \sec.^4 A}}.$$

$$\text{Sec. } x = \frac{1}{\cos. x} = \frac{1}{\sec.^2 A} = \cos.^2 A.$$

$$\text{Cosec. } x = \frac{1}{\sin. x} = \frac{1}{\sqrt{1 - \sec.^4 A}}.$$

Solve the following problems :—

1. Given, $\frac{\sqrt{1 - \cos.^2 x}}{\sin. A} = \frac{\sin.^2 x}{\cos. A}$. Find the sin. x, cos. x, tan. x.

Since $\sqrt{1 - \cos.^2 x} = \sin. x$; $\therefore \frac{\sin. x}{\sin. A} = \frac{\sin.^2 x}{\cos. A}$.

Hence $\sin. x = \frac{\cos. A}{\sin. A} = \cot. A$;

and $\cos. x = \sqrt{1 - \sin.^2 x} = \sqrt{1 - \cot.^2 A}$;

and $\tan. x = \frac{\sin. x}{\cos. x} = \frac{\cot. A}{\sqrt{1 - \cot^2 A}}$.

2. Given, $n \cos. x = \sqrt{1 + \cos.^2 x}$. Find sin. x, cos. x, and tan. x. Square and transpose.

3. Given, $a \sin. x = \frac{b}{\sin. x}$. Find sin. x, cos. x, and tan. x.

Multiply by sin. x, and extract the square root.

4. Given, $a \sqrt{1 - \sin.^2 x} = \text{vers.}^2 x - 1$. Find sin. x, cos. x, and tan. x.

ANSWERS.

2. $\sqrt{\dfrac{n^2-2}{n^2-1}}, \sqrt{\dfrac{1}{n^2-1}}, \sqrt{n^2-2}.$

3. $\sqrt{\dfrac{b}{a}}, \sqrt{\dfrac{a-b}{a}}, \sqrt{\dfrac{b}{a-b}}.$

4. $\sqrt{1-(a+2)^2}, a+2, \dfrac{\sqrt{1-(a+2)^2}}{a+2}.$

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(9). $\dfrac{\text{Tan. A cot. A sec. A}}{1-\text{vers. A}} = \dfrac{\text{sec. A}}{\text{cos. A}}$, from (3) and (8).

$= \text{sec.}^2 \text{A}$, from (2). (Jeane's Trig., p. 10. Q. 35).

Prove the following :—

1. $\dfrac{(1-\text{vers. A})\ \text{tan. A cot A}}{\text{cos.}^2\text{ A sec.}^2\text{ A}} = \text{cos. A}.$

2. $\dfrac{\text{cos. A }(1-\text{vers. A})}{\text{cos.}^2\text{ A tan. A}} = \text{cot. A}.$

3. $\dfrac{\text{sin. A cosec. A }(1-\text{vers. A})}{\text{cos. A tan. A cot. A}} = 1.$

(10.) $\sqrt{\text{sec. A}+1}\ \sqrt{\text{sec. A}-1} = \sqrt{\text{sec.}^2\text{ A}-1}.$

$= \sqrt{\text{tan.}^2\text{A}}$. By 5.

$= \text{tan. A}.$

(From Jeane's Trig., p. 10. Q. 37.)

Because the sum and difference of any two quantities, multiplied together, is equal to the difference of their squares. Very important.

Prove the following :—

1. $\sqrt{1 - \cos. A}\sqrt{1 + \cos. A} = \frac{1}{\text{cosec. } A}$.

2. $\frac{\sin.^2 A\ (1 - \text{vers. } A)}{\sqrt{1-\cos. A}\ \sqrt{1+\cos. A}\ \tan. A \cot. A \cos. A} = \sin. A.$

3. $\frac{\tan.^3 A,\ \cot. A}{\sqrt{\text{cosec. } A-1}\ \sqrt{\text{cosec.}+1}\ \sqrt{1-\text{vers.} A} \sec. A} = \tan.^3 A.$

(11). $\text{Tan.}^2 A - \sin.^2 A = \frac{\sin.^2 A}{\cos.^2 A} - \sin.^2 A.$ By 7.

$$= \sin.^2 A\left(\frac{1}{\cos.^2 A} - 1\right).$$

$$= \sin.^2 A\ \frac{(1 - \cos.^2 A}{\cos.^2 A}.$$

$$= \frac{\sin.^2 A}{\cos.^2 A} \sin.^2 A. \quad \text{By 4.}$$

$= \tan.^2 A \sin.^2 A.$ (Jeane's Trig., p. 10. Q. 42.)

Prove the following :—

1. $\frac{\sec.^2 A - \cos.^2 A}{1 + \cos.^2 A} = \tan.^2 A.$

2. $\dfrac{\text{cosec.}^2 A - \sin.^2 A}{1 + \sin.^2 A} = \cot.^2 A.$

3. $\dfrac{\sec.^2 A \cos. A}{\tan. A \cot. A} - \cos. A = \tan. A \sin. A.$

(12). Express all the trigonometrical functions of A in terms of the cosec. A. (Jeane's Trig. p. 11. Q. 51).

$$\text{Sin. } A = \frac{1}{\text{cosec. } A}. \quad \text{By 1.}$$

$$\text{Cos. } A = \sqrt{1 - \sin.^2 A}. \quad \text{By 4.}$$

$$= \sqrt{1 - \frac{1}{\text{cosec.}^2 A}}.$$

$$= \frac{\sqrt{\text{cosec.}^2 A - 1}}{\text{cosec. } A}.$$

$$\text{Tan. } A = \frac{\sin. A}{\cos. A}. \quad \text{By 7.}$$

$$= \frac{1}{\text{cosec. } A} \times \frac{\text{cosec. } A}{\sqrt{\text{cosec.}^2 A - 1}} = \frac{1}{\sqrt{\text{cosec.}^2 A - 1}}.$$

$$\text{Cot. } A = \frac{1}{\tan. A}. \quad \text{By 3} = \sqrt{\text{cosec.}^2 A - 1}.$$

$$\text{Sec. } A = \frac{1}{\cos. A}. \quad \text{By 2} = \frac{\text{cosec. } A}{\sqrt{\text{cosec.}^2 A - 1}}.$$

Vers. A $= 1 -$ cos. A. By 8.

$$= 1 - \frac{\sqrt{\text{cosec.}^2 - 1}}{\text{cosec. A}}.$$

$$= \frac{\text{cosec. A} - \sqrt{\text{cosec.}^2 - 1}}{\text{cosec. A}}.$$

Solve from 45 to 51. (Jeane's Trig., pp. 10 and 11.)

(13). Given, $\sqrt{1 - a^2}$ tan. $x = a$. Find the other trigonometrical functions of x in terms of a.

$$\therefore \text{tan. } x = \frac{a}{\sqrt{1 - a^2}}.$$

$$\text{Cos. } x = \frac{1}{\text{sec. } x}. \quad \text{By 2.}$$

$$= \frac{1}{\sqrt{1 + \text{tan.}^2 x}}. \quad \text{By 5.}$$

$$= \frac{1}{\sqrt{1 + \frac{a^2}{1 - a^2}}}.$$

$$= \frac{\sqrt{1 - a^2}}{\sqrt{1 - a^2 + a^2}} = \sqrt{1 - a^2}.$$

$$\text{Sin. } x = \sqrt{1 - \text{cos.}^2 x} = \sqrt{1 - 1 + a^2} = a.$$

$$\text{Cot. } x = \frac{1}{\text{tan. } x} = \frac{\sqrt{1 - a^2}}{a}. \quad \text{From 3.}$$

$$\text{Sec. } x = \frac{1}{\text{cos. } x} = \frac{1}{\sqrt{1 - a^2}}. \quad \text{From 2.}$$

$$\text{Cosec.} x = \frac{1}{\text{sin. } x} = \frac{1}{a}. \quad \text{From 1.}$$

Solve the following, and find all the trigonometrical functions of x.

1. $n \sec. x = \sqrt{1 + n^2}$.

2. $n \dfrac{\sqrt{1 - \cos.^2 x}}{\sin.^2 x} = \operatorname{cosec.}^2 x$.

3. $\dfrac{(1 - \operatorname{vers.} x) \sec.^2 x}{5 \tan.^2 x \cot. x} = 1$.

4. $\dfrac{\sqrt{\sec. x - 1} \quad \sqrt{\sec. x - 1}}{\tan.^2 x} = \dfrac{1}{10}$.

ANSWERS.

1. $\dfrac{1}{\sqrt{1+n^2}}, \dfrac{n}{\sqrt{1+n^2}}, \dfrac{1}{n}, n, \dfrac{\sqrt{1+n^2}}{n}, \sqrt{1+n^2}$.

2. $\dfrac{1}{n}, \dfrac{\sqrt{n^2-1}}{n}, \dfrac{1}{\sqrt{n^2-1}}, \sqrt{n^2-1}, \dfrac{n}{\sqrt{n^2-1}}, n$.

3. $\dfrac{1}{5}, \dfrac{2}{5}\sqrt{6}, \dfrac{\sqrt{6}}{12}, 2\sqrt{6}, \dfrac{5\sqrt{6}}{12}, 5$.

4. $\dfrac{10}{\sqrt{101}}, \dfrac{1}{\sqrt{101}}, 10, \dfrac{1}{10}, \sqrt{101}, \dfrac{\sqrt{101}}{10}$.

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(14). Given, $\sin.^2 x + 5 \cos.^2 x = 3$. Find all the trigonometrical functions of x. (Jeane's Trig., p. 12. Q. 63.)

Since $\sin.^2 x = 1 - \cos.^2 x$. By 4.

$$\therefore 1 - \cos.^2 x + 5 \cos.^2 x = 3.$$

Or, $4 \cos.^2 x = 2$; $\therefore 2 \cos. x = \sqrt{2}$; $\therefore \cos. x = \frac{\sqrt{2}}{2}$.

$$\text{Sin.}\ x = \sqrt{1 - \cos.^2 x} = \sqrt{1 - \frac{1}{2}} = \frac{\sqrt{2}}{2}, \text{ and}$$

$$\tan. x = \frac{\sin. x}{\cos. x} = 1.$$

The other functions are readily obtained.

Solve the following :—

1. $3 \tan.^2 x + 5 \sec.^2 x = 47.$

2. $2 \operatorname{cosec.}^2 x - 3 \sin.^2 x = \dfrac{n}{\sin.^2 x}.$

ANSWERS.

1. $\frac{\sqrt{21}}{5}, \frac{2}{5}, \frac{\sqrt{21}}{2}, \frac{2}{\sqrt{21}}, \frac{5}{2}, \frac{5}{\sqrt{21}}.$

2. $\sqrt[4]{\frac{2-n}{3}}, \sqrt{1 - \sqrt{\frac{2-n}{3}}} = \sin. x,$

$\cos. x$ respectively.

(15). Given, tan. x + cot. $x = 4$. Find all the trigonometrical functions of x.

$$\text{Since cot. } x = \frac{1}{\text{tan. } x}. \quad \text{By 3.}$$

$$\therefore \text{tan. } x + \frac{1}{\text{tan. } x} = 4.$$

Or, $\text{tan.}^2\, x - 4 \text{ tan. } x = -1$. A quadratic.

Or, $\text{tan.}^2\, x - 4 \text{ tan. } x + 4 = 3$;

$\therefore$ tan. $x - 2 = \pm \sqrt{3}$; or, tan. $x = 2 \pm \sqrt{3}$.

(Jeane's Trig. p. 12. Q. 69.)

The other functions may be readily obtained.

Solve the following :—

1. Given, sin. x + 4 cosec. $x = 2\,n$. Find sin. x.
2. Given, sec. $x + n$ cos. $x = 2\,m$. Find sec. x.
3. Given, 4 tan. x cot. x + sin. $x + a$ cosec. $x = 0$. Find sin. x.
4. Given, tan x + 12, cot. $x = 7$. Find sec. x.

(From Newth.)

ANSWERS.

1. $n \pm \sqrt{n^2 - 4}$.
2. $m \pm \sqrt{m^2 - n}$.
3. $-2 \pm \sqrt{4 - a}$.
4. $\pm \sqrt{17}$; or $\pm \sqrt{10}$.

(16). Given, sin. x (sin. x — cos. x) = m. Find sin. x. (Jeane's Trig., p. 12. Q. 70.)

$$\text{Sin.}^2 x - \text{sin. } x \text{ cos. } x = m;$$

$$\text{or, sin. } x \sqrt{1 - \text{sin.}^2 x} = \text{sin.}^2 x - m;$$

$$\therefore \text{sin.}^2 x - \text{sin.}^4 x = \text{sin.}^4 x + m^2 - 2m \text{ sin.}^2 x;$$

$$\therefore \text{sin.}^4 x - \frac{2m+1}{2} \text{ sin.}^2 x = -\frac{m^2}{2}.$$

From which quadratic we have—

$$\text{sin. } x = \pm \frac{1}{2}\sqrt{2m + 1 \pm \sqrt{1 + 4m - 4m^2}}.$$

Solve the following:—

1. Tan. x (tan. x — cot. x) = n — 2 — 2 tan. x. Find sin. x.
2. Cos. x (cos. x + sin. x) = 1. Find tan. x.
3. Sin. x + cos. x = $\frac{7}{5}$ (from Newth). Find cos. x.
4. Sin x — cos. x = a. Find sin. x and cos. x.

ANSWERS.

1. $\dfrac{\pm\sqrt{n} - 1}{\sqrt{n + 2 \pm 2\sqrt{n}}}$.

2. 1. 3. $\frac{4}{5}$; or $\frac{3}{5}$.

4. $\text{Cos. } x = \dfrac{-a \pm \sqrt{2 - a^2}}{2}$.

$\text{Sin. } x = \dfrac{a \pm \sqrt{2 - a^2}}{2}$.

(17). Given, $\left.\begin{array}{r}\sin. x + \sin. y = m \quad (1).\\ \sin. x, \sin. y = n \quad (2).\end{array}\right\}$ Find $\sin. x.$ $\sin. y.$

Square (1). $\text{Sin.}^2 x + \sin.^2 y + 2 \sin. x \sin. y = m^2$;

$\therefore 4 \sin. x \sin. y = 4 n.$

$\therefore \sin.^2 x + \sin.^2 y - 2 \sin. x, \sin. y = m^2 - 4 n$;

$\therefore \sin. x - \sin. y = \pm \sqrt{m^2 - 4 n}$. Add this to (1).

$$\therefore \sin. x = \frac{m}{2} \pm \frac{1}{2}\sqrt{m^2 - 4 n};$$

$$\therefore \sin. y = \frac{m}{2} \mp \frac{1}{2}\sqrt{m^2 - 4 n}.$$ (Jeane's Trig. p. 12. Q. 71.)

Solve the following:—

1. Given, $\left.\begin{array}{r}\cos. x + \cos. y = a.\\ \cos. x \quad \cos. y = b.\end{array}\right\}$ Find $\cos. x$ and $\cos. y$.

2. Given, $\left.\begin{array}{r}a \sec. x + b \operatorname{cosec}. y = c.\\ \sec. x \quad \operatorname{cosec}. y = d.\end{array}\right\}$ Find $\sec. x$ and $\operatorname{cosec}. y$.

3. Given, $\left.\begin{array}{r}3 \tan. x + 4 \cot. y = 13.\\ \tan. x \quad \cot. y = 3.\end{array}\right\}$ Find $\tan. x$ and $\cot. y$.

ANSWERS.

1. $\text{Cos. } x = \dfrac{a \pm \sqrt{a^2 - 4 b}}{2}$

$\text{Cos. } y = \dfrac{a \mp \sqrt{a^2 - 4 b}}{2}$.

2. $$\text{Sec. } x = \frac{c \pm \sqrt{c^2 - 4abd}}{2a}.$$

$$\text{Cosec. } y = \frac{c \mp \sqrt{c^2 - 4abd}}{2b}.$$

3. Tan. $x = 3$.

Cot. $y = 1$.

---

(18). Given, $\left.\begin{aligned} \sin. x + \sin. y &= a \ (1). \\ \cos.^2 x - \cos.^2 y &= b^2 \ (2). \end{aligned}\right\}$ Find sin. x and sin. y.

From Newth.

From (2). $1 - \sin.^2 x - 1 + \sin.^2 y = b^2$;

$\therefore \sin.^2 y - \sin.^2 x = b^2$. (3). Divide (3) by (1).

$$\text{Sin. } y - \sin. x = \frac{b^2}{a}. \quad \text{Add this to (1).}$$

$$\text{Sin. } y = \frac{b^2}{2a} + \frac{a}{2} = \frac{b^2 + a^2}{2a};$$

$$\therefore \text{Sin. } x = a - \frac{b^2}{2a} - \frac{a}{2} = \frac{a^2 - b^2}{2a}.$$

Solve the following :—

1. Given, $\left.\begin{aligned} \cos. x + \cos. y &= a. \\ \sin.^2 x - \sin.^2 y &= b^2. \end{aligned}\right\}$ Find cos. x and cos. y.

2. Given, $\left.\begin{aligned} \tan. x + \tan. y &= a. \\ \sec.^2 x - \sec.^2 y &= b. \end{aligned}\right\}$ Find tan. x and tan. y.

3. Given, $\left.\begin{aligned} \cot. x - \tan. y &= n. \\ \cot.^2 x - \tan.^2 y &= m. \end{aligned}\right\}$ Find cot. x and tan. y.

ANSWERS.

1. $\dfrac{a^2 - b^2}{2a}$, $\dfrac{a^2 + b^2}{2a}$.

2. $\dfrac{a^2 + b}{2a}$, $\dfrac{a^2 - b}{2a}$.

3. $\dfrac{m + n^2}{2n}$, $\dfrac{m - n^2}{2n}$.

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(19). Given, $\left.\begin{array}{l} \sin. x = a \sin. y \;(1). \\ \tan. x = b \tan. y \;(2). \end{array}\right\}$ Find $\cos. x$ and $\cos. y$.

(From Newth.)

Divide (1) by (2) ; $\therefore \dfrac{\sin. x}{\tan. x} = \dfrac{a}{b} \dfrac{\sin. y}{\tan. y}$;

$$\text{or, } \cos. x = \frac{a}{b} \cos. y.$$

Hence $\sin.^2 x = a^2 \sin.^2 y$; and $\cos.^2 x = \dfrac{a^2}{b^2} \cos.^2 y$.

$$\therefore \sin.^2 x + b^2 \cos.^2 x = a^2 (\sin.^2 y + \cos.^2 y) = a^2.$$

$$1 - \cos.^2 x + b^2 \cos.^2 x = a^2 ; \;\therefore \cos. x = \sqrt{\frac{a^2 - 1}{b^2 - 1}};$$

$$\therefore \cos. y = \frac{b}{a} \cos. x = \frac{b}{a} \sqrt{\frac{a^2 - 1}{b^2 - 1}}.$$

Solve the following :—

1. Given, $\left.\begin{aligned}\cos. x &= a \cos. y.\\ \cot. x &= b \cot. y.\end{aligned}\right\}$ Find cos. x and cos. y.

2. Given, $\left.\begin{aligned}\sec. x &= a \sec. y.\\ \text{cosec.}\, x &= b\, \text{cosec.}\, y.\end{aligned}\right\}$ Find cos. x and cos. y.

3. Given, $\left.\begin{aligned}\tan. x &= a \tan. y.\\ \sec. x &= b \sec. y.\end{aligned}\right\}$ Find tan. x, tan. y.

ANSWERS.

1. $\sqrt{\dfrac{a^2 - b^2}{1 - b^2}}, \quad \dfrac{1}{a}\sqrt{\dfrac{a^2 - b^2}{1 - b^2}}.$

2. $\sqrt{\dfrac{b^2 - 1}{b^2 - a^2}}, \quad a\sqrt{\dfrac{b^2 - 1}{b^2 - a^2}}.$

3. $a\sqrt{\dfrac{1 - b^2}{b^2 - a^2}}, \quad \sqrt{\dfrac{1 - b^2}{b^2 - a^2}}.$

---

(20). Given tan. $\theta = 3$. Find the value of

$$\frac{\sin. \theta + \cos. \theta}{\tan. \theta + \cot. \theta}.$$

$$\text{Since } \sin. \theta = \frac{1}{\text{cosec.}\, \theta} = \frac{1}{\sqrt{1 + \cot.^2 \theta}}$$

$$= \frac{\tan. \theta}{\sqrt{1 + \tan.^2 \theta}} = \frac{3}{\sqrt{10}}.$$

$$\text{Cos. } \theta = \sqrt{1 - \text{sin.}^2 \theta} = \sqrt{1 - \frac{9}{10}} = \frac{1}{\sqrt{10}}.$$

$$\therefore \frac{\text{sin. } \theta + \text{cos. } \theta}{\text{tan. } \theta + \text{cot. } \theta} = \frac{\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}}}{3 + \frac{1}{3}} = \frac{3}{\sqrt{10}} \cdot \frac{4}{10}$$

$$= \frac{3\sqrt{10}}{25}.$$

Solve the following :—

1. Given, sin. $\theta = \frac{1}{3}$. Find $\frac{\text{tan. } \theta + \text{cot. } \theta}{1 - \text{vers. } \theta}$. Ans. $\frac{27}{8}$.

2. Given, sec. $\theta = 4$. Find $\frac{1}{\text{cosec. } \theta} + \frac{1}{\text{tan. } \theta} - \text{sin. } \theta$.

Ans. $\frac{1}{\sqrt{15}}$.

3. Given, tan. $\theta = 2$. Find $\frac{\text{sin. } \theta - 2 \text{ cos. } \theta + 1}{\text{sec. } \theta + \text{cosec. } \theta - 1}$.

Ans. $\frac{2}{3\sqrt{5} - 2}$.

(21). Find the circular measure of 23 degrees.

The circular measure of 180 deg. $= 3 \cdot 1416$.

" " 1 " $= \frac{3 \cdot 1416}{180}$.

" " 23 " $= \frac{3 \cdot 1416 \times 23}{180}$

$= \cdot 40142$.

EXAMPLES.

1.	Find the circular measure of	120 deg.		Ans.	2·0944.
2.	,, ,,	74	,,	,,	1·2915.
3.	,, ,,	140	,,	,,	2·4434.
4.	,, ,,	56	,,	,,	·97739.
5.	,, ,,	70	,,	,,	1·2217.
6.	,, ,,	132	,,	,,	2·3038.

(22). Find the degrees in an angle whose circular measure is 1·42.

The degrees in an angle whose circular measure is 3·1416 = 180 degrees.

The degrees in an angle whose circular measure is 1

$$= \frac{180}{3 \cdot 1416} = 57 \cdot 2957.$$

The degrees in an angle whose circular measure is 1·42

$$= \frac{180 \times 1 \cdot 42}{3 \cdot 1416} = 81^\circ\ 21'.$$

EXAMPLES.

1. Find the degrees in an angle whose circular measure is 2·5. Ans. 143° 14′.
2. Find the degrees in an angle whose circular measure is 1·5. Ans. 85° 56′.
3. Find the degrees in an angle whose circular measure is 4.3. Ans. 246° 22′.
4. Find the degrees in an angle whose circular measure is 5. Ans. 286° 28′.
5. Find the degrees in an angle whose circular measure is 6·2. Ans. 355° 13′.
6. Find the degrees in an angle whose circular measure is 2. Ans. 114° 35′.

EXAMPLES UNDER THE FOREGOING ARTICLES.

1. Transform the fraction $\dfrac{1}{\sin. A, \operatorname{cosec.} A \sqrt{1 - \cos.^2 A}}$ to an integral form.

Ans. cosec. A.

2. If $\sin. x = n^2 \operatorname{cosec.} x$. Find sin. x, cos. x, tan. x.

$$\text{Sin. } x = \pm n;\ \cos. x = \sqrt{1-n^2};\ \tan x = \pm \frac{n}{\sqrt{1-n^2}}.$$

3. Given, $\cot. x = \tan. A$. Find sin. x, cos. x, tan. x.

$$\text{Sin. } x = \frac{\cot. A}{\sqrt{1+\cot.^2 A}};\ \cos. x = \frac{1}{\sqrt{1+\cot.^2 A}};\ \tan. x = \cot. A.$$

4. Given, $\sin. x = \frac{1}{2} \sqrt{1-\sin.^2 x}$, to find sin. x, cos. x, tan. x.

$$\text{Sin. } x = \frac{1}{\sqrt{5}};\ \cos. x = \frac{2}{\sqrt{5}};\ \tan. x = \frac{1}{2}.$$

5. Prove that $\cot.^2 A - \cos^2 A = \cot.^2 A, \cos.^2 A$.

6. Given, $2 \sec. x + \operatorname{cosec.} x = \dfrac{1}{\sin. x, \cos. x}$. Find sin. x.

$$\text{Sin. } x = \frac{4}{5}.$$

7. Given, $\left.\begin{aligned} \sin. x &= n \cos. y. \\ \cos. x &= m \sin. y. \end{aligned}\right\}$ Find sin. x and sin. y.

$$\text{Sin. } x = n \sqrt{\frac{m^2 - 1}{m^2 - n^2}},\ \sin. y = \sqrt{\frac{1 - n^2}{m^2 - n^2}}.$$

8. Transform $\dfrac{\sin. A}{(1 - \text{vers. } A)\sqrt{1 - \sin.^2 A\,(1 + \tan.^2 A)}}$ to an integral form.

Ans. sin. A.

9. Given, $1 - \text{vers. } x = 5\cos.^2 x$. Find sin. x, cos. x, tan. x.

$$\text{Sin. } x = \frac{2}{5}\sqrt{6};\ \cos. x = \frac{1}{5};\ \tan. x = 2\sqrt{6}.$$

10. Given, $\dfrac{\text{cosec. } x}{\sec. x} = 2$. Find sin. x, cos. x, tan. x.

$$\text{Sin. } x = \frac{1}{\sqrt{5}},\ \cos. x = \frac{2}{\sqrt{5}},\ \tan. x = \frac{1}{2}.$$

11. Given, sec. x = 3 tan. x. Find sin. x, cos. x, tan. x.

$$\text{Sin. } x = \frac{1}{3},\ \cos. x = \frac{2}{3}\sqrt{2},\ \tan. x = \frac{1}{2\sqrt{2}}.$$

12. Given, $\sin.^2 x - \cos.^2 x = \dfrac{1}{4}(\sin. x - \cos. x)$. Find sin. x, cos. x.

$$\text{Sin. } x = \frac{1 \pm \sqrt{31}}{8};\ \cos. x = \frac{1 \pm \sqrt{31}}{8};$$

$$\text{or, } \frac{\sqrt{2}}{2};\ \text{or, } \frac{\sqrt{2}}{2}.$$

13. Given, $\left.\begin{array}{l}\tan. x = n \tan. y. \\ \cot. x = m \tan. y.\end{array}\right\}$ Find tan. x, tan. y.

$$\frac{n}{\sqrt{n\,m}}\quad \frac{1}{\sqrt{n\,m}}.$$

14. Prove $\dfrac{\sec.^3 A - \text{cosec.}^3 A}{(\sin. A - \cos. A)\,(1 + \sin. A \cos. A)} = \sec.^3 A\ \text{cosec.}^3 A.$

15. Transform $\frac{1}{\text{sin. A sec. A tan. A cos. A}}$ to an integral form.

Cosec.^{2} A cos. A.

16. Given, tan $x = 25$ cot. x. Find sin. x, cos. x, tan. x.

Sin. $x = \pm \frac{5}{\sqrt{26}}$ cos. $x = \frac{1}{\sqrt{26}}$ and tan. $x = \pm 5$.

17. Given, cosec. A $= n$ sec. x. Find sin. x, cos. x, tan. x.

Sin. $x = \sqrt{1 - n^2 \sin^2 A}$; cos. $x = n$ sin. A,

$$\tan. x = \frac{\sqrt{1 - n^2 \sin.^2 A}}{n \sin. A}.$$

18. Given, sin. x cos. $x = \frac{1}{2}$. Find sin. x, cos. x, tan. x.

Sin. $x = \pm \frac{\sqrt{2}}{2}$ cos. $x = \pm \frac{\sqrt{2}}{2}$ tan. $x = \pm 1$.

19. Given, $\left.\begin{array}{r} \sin. x + \cos. y = \frac{1}{2} \\ \sin. x \cos. y = \frac{1}{25} \end{array}\right\}$ Find sin. x and cos. y.

Sin. $x = \frac{2}{5}$; cos. $y = \frac{1}{10}$;

or, $\frac{1}{10}$; or, $\frac{2}{5}$.

20. Prove $\frac{(\sin.^3 A + \cos.^3 A)}{(\sec. A + \text{cosec. } A)(1 - \sin. A . \cos. A)}$

$= \cos. A \sin. A.$

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21. Prove sec.^{2} A + cosec.^{2} A = sec.^{2} A . cosec.^{2} A.

22. Find the value of $\dfrac{a \sin. \theta + b \cos. \theta}{a \sin. \theta - b \cos. \theta}$,

where $\tan. \theta = \dfrac{a}{b}$. (From Colenso.)

Ans. $\dfrac{a^2 + b^2}{a^2 - b^2}$.

23. Eliminate θ from the equations

$$m \sec. \theta - \tan. \theta = n \operatorname{cosec}. \theta + \cot. \theta = 1.$$

$m^2 + n^2 = 2$. (From Colenso.)

24. If $\sec. A + \operatorname{cosec}. A = m$, and $\sec. A - \operatorname{cosec}. A = n$,

show that $\tan. A = \dfrac{m + n}{m - n}$,

and, $(m^2 - n^2)^2 = 8 (m^2 + n^2)$. (From Colenso.)

(B.)

$$\text{Sin. } A = \cos.\ (90 - A) = \sin.\ (180 - A) \qquad 1.$$

$$\text{Cos. } A = \sin.\ (90 - A) = -\cos.\ (180 - A) \qquad 2.$$

From these equations corresponding expressions for tan. A, cot. A, &c., can be readily obtained by the formulæ (A).

$$\text{Sin. } A = -\sin.\ (-A) \quad . \quad . \quad . \quad 3.$$

$$\text{Cos. } A = \cos.\ (-A) \quad . \quad . \quad . \quad . \quad 4.$$

Two principles only are necessary to amplify these relations to an unlimited extent:—

1st. By substitution of any angle for A;

2nd. By adding, or subtracting, four right angles, or 360 degrees, to or from A, the trigonometrical ratios, sin. A, cos. A, &c., are not altered in value.

(1). Illustration of 1st principle.

In (1), substitute $(90 - A)$ for A;

$$\therefore \sin. (90 - A) = \sin. (180 - 90 + A).$$

$$\text{or, } \cos. A = \sin. (90 + A). \quad \text{Important.}$$

In (2), substitute $(90 - A)$ for A;

$$\therefore \cos. (90 - A) = -\cos. (180 - 90 + A).$$

$$\text{or, } \sin. A = -\cos. (90 + A). \quad \text{Important.}$$

In (1), substitute $-A$ for A;

$$\therefore \sin. (-A) = \sin. (180 + A).$$

$$\text{or, } \sin. A = -\sin. (180 + A). \quad \text{Important.}$$

In (2), substitute $-A$ for A;

$$\therefore \cos. -A = -\cos. (180 + A).$$

$$\text{or, } \cos. A = -\cos. (180 + A). \quad \text{Important.}$$

(2). Illustration of 2nd principle.

Add successively 360, or 2π, to the angle in (1);

$$\begin{aligned} \therefore \sin. A &= \sin. (2\pi + A) = \sin. (4\pi + A) \\ &= \sin. (6\pi + A) = \sin. (2n\pi - A), \end{aligned}$$

where n is any whole number taken from the series 0, 1, 2, 3, &c.

Add successively 2π to the angle in (3);

$$\begin{aligned} \therefore \sin. A &= -\sin. (2\pi - A) = -\sin. (4\pi - A) \\ &= -\sin. (6\pi - A) = -\sin. (2n\pi - A). \end{aligned}$$

Hence $\sin. A = \pm \sin. (2n\pi \pm A)$ . . . (1).

Add successively 2π to the angles in (2) and (4);

$$\therefore \cos. A = \cos. (2\pi + A) = \cos. (4\pi + A)$$
$$= \cos (6\pi + A) = \cos. (2n\pi + A);$$

$$\text{and } \cos. A = \cos. (2\pi - A) = \cos. (4\pi - A)$$
$$= \cos. (6\pi - A) = \cos. (2n\pi - A).$$

Hence $\cos. A = \cos. (2n\pi \pm A)$ . . . . (2).

From (1) and (2) we have $\tan. A = \pm \tan. (2n\pi \pm A)$.
(Jeane's Trig., p. 31. Q. 94.)

(3). Find the sine and cosine of ($\pm$ 54) degrees from the tables.

$$\left.\begin{array}{l} \text{Sin. } 54 = \sin. (90 - 36) = \cos. 36 = \cdot 8090170 \\ \text{Cos. } 54 = \cos. (90 - 36) = \sin. 36 = \cdot 5877853 \end{array}\right\} \text{By (1) and (2) (B).}$$

$$\left.\begin{array}{l} \text{Sin. } (-54) = -\sin. 54 = -\cos. 36 = -\cdot 8090170 \\ \text{Cos. } -54 = \cos. 54 = \cdot 5877153. \end{array}\right\} \text{By (3) and (4) (B).}$$

EXAMPLES.

1. Find the sine and cosine of 84. Ans. ·99452 and ·10452.
2. „ „ 75. „ ·96592 and ·25881.
3. „ „ 85. „ ·99619 and ·08715.

(4). Find the sine and cosine of $\pm$ 106 degrees from the tables.

$$\text{Sin. } 106 = \sin. (180 - 74) = \sin. 74 = \sin. (90 - 16)$$
$$= \cos. 16 = \cdot 96126.$$

$$\text{Cos. } 106 = \cos. (180 - 74) = -\cos. 74 = -\cos. (90 - 16)$$
$$= -\sin. 16 = -\cdot 27563.$$

These equations are obvious from (1) and (2) in (B).

$$\left.\begin{array}{l}\text{Sin.} - 106 = -\text{ sin. } 106 = -\text{ cos. } 16. \\ \text{Cos.} - 106 = \text{cos. } 106 = -\text{ sin. } 16.\end{array}\right\} \text{From (3) and (4) in (B).}$$

EXAMPLES.

1.	Find the sine and cosine of	± 124.	Ans.	cos. 34,	—sin. 34,	—cos. 34,	—sin. 34.
2.	„	± 138.	„	sin. 42,	—cos. 42,	—sin. 42,	—cos. 42.
3.	„	± 172.	„	sin. 8,	—cos. 8,	—sin. 8,	—cos. 8.

(5). Find the sine and cosine of ± 408 degrees from the tables.

$$\text{Sin. } 408 = \text{sin. } (360 + 48) = \text{sin. } 48 = \text{sin. } (90 - 42) = \text{cos. } 42.$$

$$\text{Cos. } 408 = \text{cos. } (360 + 48) = \text{cos. } 48 = \text{cos. } (90 - 42) = \text{sin. } 42.$$

These equations follow from the 2nd principle, viz., adding or subtracting 360 to any angle without altering the sine or cosine.

$$\left.\begin{array}{l}\text{Sin. } (-408) = -\text{ sin. } 408 = -\text{ cos. } 42. \\ \text{Cos. } (-408) = \text{cos. } 408 = \text{sin. } 42.\end{array}\right\} \text{From (3) and (4) (B).}$$

EXAMPLES.

1.	Find the sine and cosine of	± 372.	Ans.	sin. 12,	cos. 12,	—sin. 12,	cos. 12.
2.	„	± 432.	„	cos. 18,	sin. 18,	—cos. 18,	sin. 18.
3.	„	± 472.	„	cos. 22,	—sin. 22,	—cos. 22,	—sin. 22.
4.	„	± 624.	„	—cos. 6,	—sin. 6,	cos. 6,	—sin. 6.
5.	„	± 840.	„	cos. 30,	—sin. 30,	—cos. 30,	—sin. 30.
6.	„	± 1056.	„	—sin. 24,	cos. 24,	sin. 24,	cos. 24.
7.	„	± 5067.	„	sin. 27,	cos. 27,	—sin. 27,	cos. 27.

8. Find the sine and cosine of	$\pm$ 1307.	,,	$-$cos. 43,	$-$sin. 43,	cos. 43,	$-$sin. 43.
9. ,,	$\pm$ 704.	,,	$-$sin. 16,	cos. 16,	sin. 16,	cos. 16.
10. ,,	$\pm$ 805.	,,	cos. 5,	sin. 5,	$-$cos. 5,	sin. 5.
11. ,,	$\pm$ 506.	,,	sin. 34,	$-$cos. 34,	$-$sin. 34,	$-$cos. 34.
12. ,,	$\pm$ 1405.	,,	$-$sin. 35,	cos. 35,	sin. 35,	cos. 35.

(6). If $\cos. (90 - x) = \dfrac{\sin. (180 - A)}{\cos. A}$, then $\sin. x = \tan. A$. (Jeane's Trig., p. 28. Q. 75.)

Since $\cos. (90 - x) = \sin. x$, and $\sin. (180 - A) = \sin. A$. From (1) (B).

Then $\sin. x = \dfrac{\sin. A}{\cos. A} = \tan. A$.

N.B. $\pi = 180$ degrees, except when circular measure is referred to.

1. If $\sec. (90 - x) = \dfrac{\operatorname{cosec.} (180 - A)}{\sin. A}$. Show that

$$\sin. x = \sin.^2 A.$$

2. If $\tan. (90 - x) \cos. (90 + x) = \dfrac{\cos. (180 + A)}{\tan. A}$;

$$\therefore \cos. x = \frac{\cos.^2 A}{\sin. A}.$$

3. If $\dfrac{\sec. (90 - x)}{\tan. (90 - x)} \cos. (90 + x) = - \dfrac{\tan. (180 + A)}{\operatorname{cosec.} (180 - A)}$;

$$\therefore \tan. x = \frac{\sin.^2 A}{\cos. A}.$$

4. Prove $\dfrac{\cos.(90-x)\sin.(90-x)}{\cos.(90+x)\sin.(90+x)}$

$$=\frac{\cos.(180-A)\sin.(180-A)}{\cos.(180+A)\sin.(180+A)}.$$

(7). If

$$\frac{\sec.(\pi-x)\cos.\left(\frac{\pi}{2}+A\right)}{\sec.(\pi-A)}=\tan.\left(\frac{\pi}{2}-A\right)\sin.\left(\frac{\pi}{2}-x\right);$$

$\therefore \sec. x = \text{cosec.} A \times \sqrt{-1}$. (Jeane's Trig., p.28. Q.78.)

By (A). $\dfrac{\cos.(\pi-A)\cos.\left(\frac{\pi}{2}+A\right)}{\cos.(\pi-x)}=\cot. A \cos. x.$

Since $\cos.(\pi-A)=-\cos. A$ $\cos.\left(\frac{\pi}{2}+A\right)=-\sin. A$

and $\cos.(\pi-x)=-\cos. x$;

$$\therefore -\frac{\cos. A\sin. A}{\cos. x}=\frac{\cos. A\cos. x}{\sin. A};\ \therefore \cos.^2=-\sin.^2 A;$$

or, $\sec.^2 x=-\text{cosec.}^2 A$; $\therefore \sec. x=\text{cosec.} A\times\sqrt{-1}$.

1. If $\dfrac{\sin.(90-A)\cos.(90+A)}{\sec.(180-A)\ \text{cosec.}(90+A)}=\dfrac{\cot.(180+x)}{\tan.(180-x)}$

$\therefore \cot.^2 x=\cos.^3 A\sin. A.$

2. If $\dfrac{\sin.(180+x)\cos.(180-x)}{\tan.(90-x)\cot.\left(\frac{\pi}{2}+x\right)}=\dfrac{\cot.(90-A)}{\tan.(90-A)}.$

Prove $2\sin. x=\sqrt{1-2\tan.^2 A}+\sqrt{1+2\tan.^2 A}$.

3. If $\dfrac{\tan.(180-x)\ \tan.\left(\frac{\pi}{2}+x\right)}{\sin.\left(\frac{\pi}{2}-x\right)\ \cos.(90+x)} = \dfrac{\sec.(90+A)}{\operatorname{cosec.}(90-A)}$;

$\therefore \sin. x \cos. x = \tan. A.$

4. If $\dfrac{\sin.(90-x)\ \cos.(90+x)\ \sec.(90+x)}{\cot.(180-A)\ \cot.(180+A)\ \cos. x}$.

$= -1 + \operatorname{vers.}(\pi - x)\ ;\ \therefore \cos. x = -\tan.^2 A.$

(8). Find sin. x in

$$\sin.(\pi - x)\ \left\{1 - \operatorname{vers.}\left(\frac{\pi}{2}+x\right)\right\}$$
$$= \cos.^2(90-x) + 4\cos.(90-x) + 8.$$

Since $\sin.(\pi - x) = \sin. x$, and

$$1 - \operatorname{vers.}\left(\frac{\pi}{x}+x\right) = \cos.\left(\frac{\pi}{2}+x\right) = -\sin. x\,;$$

$$\therefore -\sin.^2 x = \sin.^2 x + 4\sin. x + 8\,;$$

$$\text{or, } \sin.^2 x + 2\sin. x = -4\,;$$

$$\text{or, } \sin.^2 x + 2\sin. x + 1 = -3\,;\ \therefore \sin. x = -1 \mp \sqrt{-3}.$$

(Work from 85 to 90 in Jeane's Trig., p. 31.)

(9). Prove $\sin. A = \sin. \left\{n\pi + (-1)^n A\right\}$ where n is any whole number from the series 0, 1, 2, 3, &c. (Colenso's Trig., p. 52.)

By adding successively 2π to Eq. (1) (B), we have,

$$\sin. A = \sin.(2\pi + A) = \sin.(4\pi + A) = \sin.(n\pi + A),$$

where n is even.

Again, sin. $A = \sin. (\pi - A)$;

$\therefore \sin. A = \sin. (2\pi + \pi - A) = \sin. (4\pi + \pi - A)$

$= \sin. (n\pi - A)$, where n is odd.

Both of these results may be expressed by

$\sin. A = \sin. \left\{ n\pi + (-1)^n A \right\}$ where n is any number taken from the series 0, 1, 2, 3, 4, &c.

Since $(-1)^n$ is positive when n is even, and negative when n is odd.

Prove from 91 to 100. (Jeane's Trig., p. 31.)

(10). Find all the roots of sin. A = 0).

Since $\sin. A = \sin. \left\{ n\pi + (-1)^n A \right\} = 0$.

Then every angle determined from $n\pi + (-1)^n A = 0$ will be a root of the given equation.

Hence $A = -\dfrac{n \cdot \pi}{(-1)^n}$, where n must be taken successively, $0, \pm 1, \pm 2, \pm 3$, &c.

Hence $0, \pm \pi, \pm 2\pi, \pm 3\pi$, &c., are the roots of the equation $\sin. A = 0$.

Therefore, $\sin. A = A (\pi^2 - A^2)(2^2\pi^2 - A^2)(3^2\pi^2 - A^2)$ &c.,

$$= C\,A\left(1 - \frac{A^2}{\pi^2}\right)\left(1 - \frac{A^2}{2^2\,\pi^2}\right)\left(1 - \frac{A^2}{3^2\,\pi^2}\right) \text{ \&c.,}$$

where C is independent of A, and is evidently the limit of $\dfrac{\sin. A}{A}$, which is unity.

$$\therefore \sin. A = A\left(1 - \frac{A^2}{\pi^2}\right)\left(1 - \frac{A^2}{2^2\pi^2}\right)\left(1 - \frac{A^2}{3^2\pi^2}\right)\text{\&c.}$$

1. Prove that

$$\cos. A = \left(1 - \frac{2^2 A^2}{\pi^2}\right)\left(1 - \frac{2^2 A^2}{3^2\pi^2}\right)\left(1 - \frac{2^2 A^2}{5^2\pi^2}\right), \text{\&c.}$$

Observe, when $\cos. A = 0$; $\therefore A = \frac{\pi}{2}$.

Again, $\cos. A = (\cos. 2n\pi \pm A)$.

If $A = \frac{\pi}{2}$, $\therefore \sin. A = 1$, and $\frac{A^2}{\pi^2} = \frac{1}{4}$;

$$\therefore 1 = \frac{\pi}{2}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right)\left(1 - \frac{1}{8^2}\right)$$

&c., to infinity.

$$= \frac{\pi}{2}\,\frac{(2^2-1)}{2^2}\,\frac{(4^2-1)}{4^2}\,\frac{(6^2-1)}{6^2}, \text{\&c.}$$

$$\therefore \frac{\pi}{2} = \frac{2^2}{1\cdot 3} \times \frac{4^2}{3\cdot 5} \times \frac{6^2}{5\cdot 7}, \text{\&c.}\ \frac{(2n)^2}{(2n-1)(2n+1)}.$$

This is the celebrated theorem of Wallis for the calculation of π. In this article the circular measure is used for the angle A. (See Jeane's Trig., p. 107.)

EXAMPLES ON THE FOREGOING ARTICLES.

1.	Find the sine and cosine of	$\pm$ 69.	Ans.	cos. 21,	sin. 21,	—cos. 21,	sin. 21.
2.	,,	$\pm$ 170.	,,	sin. 10,	—cos. 10,	—sin. 10,	—cos. 10.
3.	,,	$\pm$ 250.	,,	—cos. 20,	—sin. 20,	cos. 20,	—sin. 20.
4.	,,	$\pm$ 850.	,,	cos. 40,	—sin. 40,	—cos. 40,	—sin. 40.

5. Prove that $\sin. A = \cos.^2 (90 - A) \sec. (90 - A)$.

6. Find the sine and cosine of $\left(\frac{3\pi}{2} + A\right)$. Ans. —cos. A, sin. A.

7. Find the sine and cosine of $\pm$ 72. Ans. cos. 18, sin. 18, —cos. 18, sin. 18.

8. „ $\pm$ 154. „ sin. 26, —cos. 26, —sin. 26, —cos. 26.

9. „ $\pm$ 270. „ — 1, 0, 1, 0.

10. „ $\pm$ 720. „ 0, 1, 0, 1.

11. „ $\pm$ 800. „ cos. 10, —sin. 10, —cos. 10, —sin. 10.

12. Prove that cos. A = sin.^{2} (90 — A) cosec. (90 — A).

13. Find the sine and cosine of (90 + A). Ans. cos. A, — sin. A.

14. Find the sine and cosine of $\frac{3\pi}{4}$. Ans. cos. 45, —sin. 45.

15. „ „ $\frac{2\pi}{3}$. Ans. cos. 30, —sin. 30.

16. „ „ $\left(\frac{3\pi}{2} - A\right)$. Ans. —cos. A, —sin. A.

17. Prove, $\dfrac{\left\{1 - \text{vers.}\ (90 - x)\right\}\ \text{sec.}\ (90 + x)}{\text{cos.}\ (90 - x)\ \text{cosec.}\ (180 - x)} = -1.$

(C.)

$$\text{Sin.}\ (A+B) = \text{sin. } A \text{ cos. } B + \text{cos. } A \text{ sin. } B \quad . \quad 1$$

$$\text{Sin.}\ (A-B) = \text{sin. } A \text{ cos. } B - \text{cos. } A \text{ sin. } B \quad . \quad 2$$

$$\text{Cos.}\ (A+B) = \text{cos. } A \text{ cos. } B - \text{sin. } A \text{ sin. } B \quad . \quad 3$$

$$\text{Cos.}\ (A-B) = \text{cos. } A \text{ cos. } B + \text{sin. } A \text{ sin. } B \quad . \quad 4$$

$$\text{Tan.}\ (A+B) = \frac{\text{tan. } A + \text{tan. } B}{1 - \text{tan. } A \text{ tan. } B} \quad . \quad . \quad . \quad . \quad 5$$

$$\text{Tan.}\ (A-B) = \frac{\text{tan. } A - \text{tan. } B}{1 + \text{tan. } A \text{ tan. } B} \quad . \quad . \quad . \quad . \quad 6$$

Sin. $0 = 0$, and cos. $0 = 1$.

Sin. $90 = 1$, and cos. $90 = 0$.

Sin. $180 = 0$, and cos. $180 = -1$.

Sin. $270 = -1$, and cos. $270 = 0$.

Sin. $360 = 0$, and cos. $360 = 1$.

Sin. $60 = \frac{1}{2}\sqrt{3}$, and cos. $60 = \frac{1}{2}$.

Sin. $45 =$ cos. $45 = \frac{1}{2}\sqrt{2}$.

Sin. $18 = \frac{1}{4}(\sqrt{5} - 1)$, and cos. $18 = \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}$

See Art. (4).

(1). Let $A = B$, then 1, 3, 5 become—

$$\text{Sin. } 2\,A = 2 \sin. A \cos. A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 1$$

$$\text{Cos. } 2\,A = \cos.^2 A - \sin.^2 A = 2 \cos.^2 A - 1 = 1 - 2 \sin.^2 A \quad 2$$

$$\text{Tan. } 2\,A = \frac{2 \tan. A}{1 - \tan.^2 A} \quad . \quad . \quad . \quad . \quad . \quad . \quad 3$$

From (1) and $\sin.^2 A + \cos.^2 A = 1$, it follows, by addition and subtraction, and extracting the square root, that—

$$2 \sin. A = \sqrt{1 + \sin. 2\,A} + \sqrt{1 - \sin. 2\,A} \quad . \quad . \quad 4$$

$$2 \cos. A = \sqrt{1 + \sin. 2\,A} - \sqrt{1 - \sin. 2\,A} \quad . \quad . \quad 5$$

From (2) it follows that, $1 - \cos. 2\,A = 2 \sin.^2 A$. 6

And $1 + \cos. 2\,A = 2 \cos.^2 A$ 7

From (3), by means of an easy quadratic equation, it follows that

$$\text{Tan. } 2\,A \tan. A = \pm \sqrt{1 + \tan.^2 2\,A} - 1 = \pm \sec. 2\,A - 1 \quad . \quad 8$$

These formulæ are true if $\frac{1}{2}A$ be substituted for A.

EXAMPLES.

1. Sec. $x = n$. Find sin. $2x$, cos. $2x$, tan. $2x$. (Jeane's Trig., p. 45. Q. 206.)

$$\text{Sin. } 2x = 2 \sin. x \cos. x = \frac{2 \sin. x}{\sec. x} = \frac{2\sqrt{1-\cos.^2 x}}{\sec. x}$$

$$= \frac{2\sqrt{1 - \frac{1}{n^2}}}{n} = \frac{2\sqrt{n^2 - 1}}{n^2}.$$

$$\text{Cos. } 2x = 2\cos.^2 x - 1 = \frac{2}{\sec.^2 x} - 1 = \frac{2 - n^2}{n^2}.$$

$$\text{Tan. } 2x = \frac{\sin. 2x}{\cos. 2x} = \frac{2\sqrt{n^2 - 1}}{2 - n^2}.$$

2. Given, tan. x. Find sin. $2x$, cos. $2x$.

3. Given, sec. $x = \frac{1+n}{1-n}$. Find sin. $2x$, cos. $2x$.

4. Given, cosec. x. Find sin. $2x$, cos. $2x$, tan. $2x$.

5. Given, cos. x. Find sec $2x$, cot. $2x$.

6. Given, sin. $30 = \frac{1}{2}$. Find sin. 15, cos. 15.

7. Given, sin. $60 = \frac{1}{2}\sqrt{3}$. Find sin. 120, cos. 120.

8. Given, sin. $45 = \frac{1}{2}\sqrt{2}$. Find sin. $22\frac{1}{2}$, cos. $22\frac{1}{2}$.

9. Given, sin. $x = \frac{1}{4}$. Find sin. $\frac{x}{2}$, cos. $\frac{x}{2}$.

10. Given, sin. $x = \frac{1}{4}$. Find sin. $2x$, cos. $2x$.

11. Given, sin. $x = \frac{1}{5}$. Find sin. $\frac{x}{2}$, cos. $\frac{x}{2}$, sin. $2x$, cos. $2x$.

12. Given, sin. $x = \frac{1}{8}$. „ „

13. Given, sin. $x = \frac{1}{10}$. „ „

14. Given, cos. $x = \frac{1}{2}$. „ „

15. Given, cos. $x = \frac{1}{3}$. „ „

16. Given, cos. $2x = \frac{1}{3}$. Find sin. x, cos. x, sin. $4x$, cos. $4x$.

17. Given, cos. $30 = \frac{1}{2}\sqrt{3}$. Find sin. 15, cos. 15.

18. Given, cos. $x = \frac{3}{4}$. Find sin. $\frac{x}{2}$, cos. $\frac{x}{2}$
sin. $2x$, cos. $2x$.

19. Given, tan. $x = 5$. Find tan. $\frac{x}{2}$, tan. $2x$.

20. Given, tan. $x = 7$. „ „

21. Given, tan. $x = \sqrt{3}$. „ „

22. Given, tan. $x = 2\sqrt{2}$. „ „

23. Prove, $\cos. 2A = \dfrac{1 - \tan.^2 A}{1 + \tan.^2 A}$,

$$\text{and, } \tan.^2 \frac{A}{2} = \frac{1 - \cos. A}{1 + \cos. A}.$$

Since $\cos. 2A = \cos.^2 A - \sin.^2 A$

$$= \frac{1}{1 + \tan.^2 A} - \frac{\tan.^2 A}{1 + \tan.^2 A} = \frac{1 - \tan.^2 A}{1 + \tan.^2 A}.$$

24. Prove $\tan. \dfrac{A}{2} = \dfrac{\sin. A}{1 + \cos. A} = \dfrac{1 - \cos. A}{\sin. A}$.

Take the value of $\tan. \dfrac{A}{2}$, and multiply numerator and denominator by $2 \sin. \dfrac{A}{2}$, and $2 \cos. \dfrac{A}{2}$, &c.

25. Prove $\cos. A = \cos.^4 \dfrac{A}{2} - \sin.^4 \dfrac{A}{2}$. Resolve into factors.

26. Prove $\cot. A + \tan. A = 2 \operatorname{cosec.} 2A$.

$$\text{Cot. } A + \tan. A = \frac{1}{\tan. A} + \tan. A = \frac{\sec.^2 A}{\tan. A} = \frac{2}{2 \sin. A \cos. A}$$

27. Prove $\sec. A \operatorname{cosec.} A = 2 \operatorname{cosec.} 2A$.

28. Prove $\dfrac{\sin. 2x}{1 + \cos. 2x} \cdot \dfrac{\cos. x}{1 + \cos. x} = \tan. \dfrac{x}{2}$.

29. Prove $\dfrac{2 \sin. A + \sin. 2A}{2 \sin. A - \sin. 2A} = \cot.^2 \dfrac{A}{2}$.

30. Prove $\dfrac{1 + \sin. A}{1 + \cos. A} = \dfrac{1}{2}\left(1 + \tan. \dfrac{A}{2}\right)^2$.

31. Prove cot. A — tan. A = 2 cot. 2 A.

32. Prove $\tan.\ \frac{A}{2} = \frac{\tan.\ A}{1 + \sec.\ A}$.

33. Prove $\text{cosec.}\ 2\ A = \frac{1 + \cot.^2 A}{2 \cot.\ A}$.

34. Prove $\pm \sec.\ A = 1 + \tan.\ A \tan.\ \frac{A}{2}$.

35. Prove $2 \text{ cosec.}\ A = \tan.\ \frac{A}{2} + \cot.\ \frac{A}{2}$.

36. Prove cot. A − cot. 2 A = cosec. 2 A.

37. Prove $\cot.\ 2\ A = \frac{\cot.^2 A - 1}{2 \cot.\ A}$.

38. Prove $\frac{\sin.\ A}{\text{vers.} A} = \cot.\ \frac{A}{2}$. (Jeane's Trig., p. 46. Q. 224.)

$$\therefore \frac{\sin.\ A}{\text{vers.} A} = \frac{2 \sin.\frac{A}{2} \cos.\frac{A}{2}}{1 - \cos.\ A} = \frac{2 \sin.\frac{A}{2} \cos.\frac{A}{2}}{2 \sin.^2 \frac{A}{2}} = \cot.\ \frac{A}{2}.$$

39. Prove $\frac{\text{vers.}\ A}{\text{vers.}\ (\pi - A)} = \tan.^2 \frac{A}{2}$. (Jeane's Trig., p. 47. Q. 230.)

Since $\frac{\text{vers.}\ A}{\text{vers.}(\pi - A)} = \frac{1 - \cos.\ A}{1 - \cos.\ (\pi - A)} = \frac{1 - \cos. A}{1 + \cos. A} = \tan.^2 \frac{A}{2}$.

40. Prove 8 cos. 2 A . cosec.^{3} 2 A
= cosec.^{3} A sec. A— cosec. A . sec.^{3} A.

$$8 \cos.\ 2\ A \text{ cosec.}^3\ 2\ A = \frac{8 \cos.\ 2\ A}{\sin.^3\ 2\ A} = \frac{\cos.^2 A - \sin.^2 A}{\sin.^3\ A \cos.^3\ A}$$

$$= \frac{1}{\sin.^3 A \cos.\ A} - \frac{1}{\sin.\ A \cos.^3 A}.$$ (Jeane's Trig.)

ANSWERS.

2. $\dfrac{2\tan. x}{1+\tan.^2 x}$, $\dfrac{1-\tan.^2 x}{1+\tan.^2 x}$.

3. $\dfrac{4\sqrt{n}\,(1-n)}{(1+n)^2}$, $2\dfrac{(1-n)^2}{(1+n)^2}-1$.

4. $\dfrac{2\sqrt{\operatorname{cosec.}^2 x-1}}{\operatorname{cosec.}^2 x}$, $\dfrac{\operatorname{cosec.}^2 x-2}{\operatorname{cosec.}^2 x}$, $\dfrac{2\sqrt{\operatorname{cosec.}^2 x-1}}{\operatorname{cosec.}^2 x-2}$.

5. $\dfrac{1}{2\cos.^2 x-1}$, $\dfrac{2\cos.^2 x-1}{2\cos. x\sqrt{1-\cos.^2 x}}$.

6. ·25881 and ·96592.

7. $\dfrac{1}{2}\sqrt{3}$, $-\dfrac{1}{2}$.

8. $\dfrac{1}{2}\sqrt{2-\sqrt{2}}$, $\dfrac{1}{2}\sqrt{2+\sqrt{2}}$.

9. $\sqrt{\dfrac{4-\sqrt{15}}{8}}$, $\sqrt{\dfrac{4+\sqrt{15}}{8}}$.

10. $\dfrac{\sqrt{15}}{8}$, $\dfrac{7}{8}$.

11. $\sqrt{\dfrac{5-2\sqrt{6}}{10}}$, $\sqrt{\dfrac{5+2\sqrt{6}}{10}}$, $\dfrac{4}{25}\sqrt{6}$, $\dfrac{23}{25}$.

12. $\dfrac{1}{4}\sqrt{8-3\sqrt{7}}$, $\dfrac{1}{4}\sqrt{8+3\sqrt{7}}$, $\dfrac{3}{32}\sqrt{7}$, $\dfrac{31}{32}$.

13. $\sqrt{\dfrac{10-3\sqrt{11}}{20}}$, $\sqrt{\dfrac{10+3\sqrt{11}}{20}}$, $\dfrac{3\sqrt{11}}{50}$, $\dfrac{49}{50}$.

14. $\pm\dfrac{1}{2}$, $\pm\dfrac{\sqrt{3}}{2}$, $\dfrac{\sqrt{3}}{2}$, $-\dfrac{1}{2}$.

15. $\dfrac{\sqrt{3}}{3}$, $\dfrac{\sqrt{6}}{3}$, $\dfrac{4\sqrt{2}}{9}$, $-\dfrac{7}{9}$.

16. $\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}, \frac{4\sqrt{2}}{9}, -\frac{7}{9}$.

17. $\frac{1}{2}\sqrt{2-\sqrt{3}}, \frac{1}{2}\sqrt{2+\sqrt{3}}$.

18. $\frac{\sqrt{2}}{4}, \sqrt{\frac{7}{8}}, \frac{3\sqrt{7}}{8}, \frac{1}{8}$.

19. $\frac{\pm\sqrt{26}-1}{5}, -\frac{5}{12}$. 20. $\frac{\pm 5\sqrt{2}-1}{7}, -\frac{7}{24}$.

21. $\frac{\pm 2-1}{\sqrt{8}}, -\sqrt{3}$. 22. $\frac{\pm 3-1}{2\sqrt{2}}, -\frac{4\sqrt{2}}{7}$.

(2). Let $B = 2A$; then 1, 3, 5, in (C) become

$$\begin{aligned}\text{Sin. } 3A &= \sin. A \cos. 2A + \cos. A \sin. 2A.\\ &= \sin. A(1-2\sin.^2 A) + 2\sin. A\cos.^2 A.\\ &= \sin. A - 2\sin.^3 A + 2\sin. A(1-\sin.^2 A).\\ &= \sin. A - 2\sin.^3 A + 2\sin. A - 2\sin.^3 A.\\ &= 3\sin. A - 4\sin.^3 A.\end{aligned}$$

(Important to remember.)

$$\begin{aligned}\text{Cos. } 3A &= \cos. A\cos. 2A - \sin. A\sin. 2A.\\ &= \cos. A(2\cos.^2 A - 1) - 2\sin.^2 A\cos. A.\\ &= 2\cos.^3 A - \cos. A - 2\cos. A(1-\cos.^2 A).\\ &= 2\cos.^3 A - \cos. A - 2\cos. A + 2\cos.^3 A.\\ &= 4\cos.^3 A - 3\cos. A.\end{aligned}$$

(Important to remember.)

$$\text{Tan. } 3A = \frac{\tan. A + \tan. 2A}{1-\tan. A\tan. 2A} = \tan. A\,.\,\frac{3-\tan.^2 A}{1-3\tan.^2 A}.$$

EXAMPLES.

1. $\text{Sin. A} = 3 \text{ sin. } \frac{A}{3} - 4 \text{ sin.}^3 \frac{A}{3}$, and

$\text{Cos. A} = 4 \text{ cos.}^3 \frac{A}{3} - 3 \text{ cos. } \frac{A}{3}$.

2. $\text{Sin. } x = \frac{1}{3}$. Find sin. $3x$, cos. $3x$. Ans. $\frac{23}{27}, \frac{10\sqrt{2}}{27}$.

3. $\text{Cos. } x = \frac{1}{5}$. Find cos. $3x$, sin. $3x$. Ans. $-\frac{71}{125}, -\frac{42\sqrt{6}}{125}$.

4. $\frac{\text{sin. } 3 A + \text{cos. } 3 A}{\text{sin. A} - \text{cos. A}} + \frac{\text{sin. } 3 A - \text{cos. } 3 A}{\text{sin. A} + \text{cos. A}} = -2.$

5. $\frac{\text{sin.}^2 3 A - \text{cos.}^2 3 A}{\text{sin.}^2 A - \text{cos.}^2 A} = 1 - 4 \text{ sin.}^2 2 A$

$$= \frac{\text{cos. } 6 A}{\text{cos. } 2 A} = 2 \text{ cos. } 4 A - 1.$$

6. $\frac{\text{sin. } 3 A - \text{cos. } 3 A}{\text{sin. A} + \text{cos. A}} - \frac{\text{sin. } 3 A + \text{cos. } 3 A}{\text{sin. A} - \text{cos. A}} = 4 \text{ sin. } 2 A.$

7. $\frac{\text{sin. } 3 A + \text{sin. A}}{\text{cos. } 3 A - \text{cos. A}} = - \text{cot. A}.$

8. $\frac{\text{sin. } 3 A + \text{sin. A}}{\text{cos. A}} + \frac{\text{cos. } 3 A - \text{cos. A}}{\text{sin. A}} = 0.$

9. $\frac{\text{sin. } 3 A + \text{cos. } 3 A}{\text{sin. A} - \text{cos. A}} + 2 \text{ sin. } 2 A + 1 = 0.$

10. $\frac{\text{sin. } 3 A + \text{sin. A}}{\text{cos. } 3 A + \text{cos. A}} = \text{tan. } 2 A.$

11. $\frac{\sin. 3 A}{\sin. A} + \frac{\cos. 3 A}{\cos. A} = 4 \cos. 2 A.$

12. $\frac{\sin. 3 A}{\sin. A} - \frac{\cos. 3 A}{\cos. A} = 2.$

13. $\cot. 3 A = \frac{\cot. A (\cot.^2 A - 3)}{3 \cot.^2 A - 1}.$

These properties are readily demonstrated by the formulæ in Art. (5), together with the formulæ in Art. (1).

(3). When the sin. 3 A is given to find the sin. A, the problem is one of considerable difficulty; being the celebrated ancient problem of trisecting the angle. From Art. (5) we have :—

$$4 \sin.^3 A - 3 \sin. A + \sin. 3 A = 0;$$

$$\text{or, } \sin.^3 A - \frac{3}{4} \sin. A + \frac{\sin. 3 A}{4} = 0 \quad . \quad . \quad (a);$$

which is a cubic equation in sin. A when sin. 3 A is given. The equation (a) is frequently used in the solution of the algebraical equation $x^3 + b\,x + c = 0$.

(4). To find the numerical value of sin. 36.

$$\text{Sin. } 36 = \cos. (90 - 36) = \cos. 54;$$

$$\text{or, } \sin. (2 \times 18) = \cos. (3 \times 18);$$

$$\therefore 2 \sin. 18 \cos. 18 = 4 \cos.^3 18 - 3 \cos. 18;$$

$$\text{or, } 2 \sin. 18 = 1 - 4 \sin.^2 18.$$

From this quadratic we obtain

$$\text{Sin. } 18 = \frac{\sqrt{5} - 1}{4}, \text{ cos. } 18 = \frac{1}{2}\sqrt{\frac{5 + \sqrt{5}}{2}}.$$

EXAMPLES.

1. Show that $\sin. 9 = \frac{1}{4}\left(\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\right)$

2. Show that $\cos. 9 = \frac{1}{4}\left(\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\right)$

3. Show that $\sin. 27 = \frac{1}{4}\left(\sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}\right)$

4. Show that $\cos. 27 = \frac{1}{4}\left(\sqrt{5+\sqrt{5}} + \sqrt{3-\sqrt{5}}\right)$

5. Show that $\cos. 36 = \sin. 54 = \frac{1}{4}\left(\sqrt{5} + 1\right)$

(5). Prove $\sin. (30+B)+\sin.(30-B)=\cos.B.$

$$\text{Sin. } (30 + B) = \sin. 30 \cos. B + \cos. 30 \sin. B$$

$$= \frac{\cos. B}{2} + \frac{\sqrt{3}}{2} \sin. B \quad . \quad . \quad . \quad (a)$$

Since $\sin. 30 = \frac{1}{2}$, and $\cos. 30 = \frac{\sqrt{3}}{2}$.

Again, $\sin. (30 - B) = \sin. 30 \cos. B - \cos. 30 \sin. B$

$$= \frac{\cos. B}{2} - \frac{\sqrt{3}}{2} \sin. B \quad . \quad . \quad . \quad (b)$$

By adding, subtracting, multiplying, and dividing (*a*) and (*b*) various theorems will follow. Thus, if we add, we shall have:—

$$\text{Sin.}\ (30 + B) + \text{sin.}\ (30 + B) = \text{cos.}\ B.$$

The following examples can be worked by the same method as the above, observing to add, subtract, multiply, &c., as indicated by the example:—

1. $\text{Cos.}\ (30 - B) - \text{cos.}\ (30 + B) = \text{sin.}\ B.$

2. $\text{Sin.}\ (30 + B)\ \text{sin.}\ (30 - B) = \text{cos.}^2\ B - \dfrac{3}{4}.$

3. $\text{Cos.}\ (30 + B)\ \text{cos.}\ (30 - B) = \text{cos.}^2\ B - \dfrac{1}{4}.$

4. $\text{Tan.}\ (60 + B)\ \text{tan.}\ (60 - B) = \dfrac{3 - \text{tan.}^2\ B}{1 - 3\ \text{tan.}^2\ B}.$

5. $\text{Tan.}\ (60 + B) - \text{tan.}\ (60 - B) = \dfrac{8\ \text{tan.}\ B}{1 - 3\ \text{tan.}^2\ B}.$

6. $\text{Sin.}\ (60 + B) - \text{sin.}\ (60 - B) = \text{sin.}\ B.$

(Jeane's Trig., p. 46. Q. 219.)

7. $\text{Cos.}\ (60 + B) + \text{cos.}\ (60 - B) = \text{cos.}\ B.$

8. $\text{Sin.}\ (60 + B)\ \text{sin}\ (60 - B) = \text{cos.}\ (30 + B)\ \text{cos.}\ (30 - B).$

9. $\text{Cos.}\ (60 + B)\ \text{cos.}\ (60 - B) = \text{sin.}\ (30 + B)\ \text{sin.}\ (30 - B).$

(See Colenso's Trig., p. 65. Q. 4.)

10. $\text{Sin.}\ (45 + B)\ \text{sin.}\ (45 - B) = \dfrac{1}{2}\ \text{cos.}\ 2\ B$

$$= \text{cos}\ (45 + B)\ \text{cos.}\ (45 - B).$$

11. $\text{Tan.}\ (45 + B)\ \text{tan.}\ (45 - B) = 1.$

12. $\dfrac{\tan.(45+B)+\tan.(45-B)}{\tan.(45+B)-\tan.(45-B)}=\text{cosec. }2\,B.$

13. $\text{Tan.}(45+B)-\tan.(45-B)=\dfrac{4\tan.B}{1-\tan.^2B}=2\tan.2\,B.$

14. $\text{Tan.}(45+B)-\cot.(45+B)=\dfrac{4\tan.B}{1-\tan.^2B}=2\tan.2\,B.$

15. $\text{Tan.}(45+B)+\cot.(45+B)=2\dfrac{1+\tan.^2B}{1-\tan.^2B}=2\sec.2\,B.$

16. $\text{Sin.}^2(45+B)+\sin.^2(45-B)=1.$

17. $\text{Sin.}^2(45+B)=\dfrac{1}{2}(1+\sin.2\,B).$

18. $\text{Sin.}^2(45-B)=\dfrac{1}{2}(1-\sin.2\,B).$

19. $\dfrac{1-\cot.^2(45+A)}{1+\cot.^2(45+A)}=\dfrac{2\tan.A}{1+\tan.^2A}=\sin.2\,A.$

(6). Shew that,

$\text{Sin.}A+\sin.(72+A)-\sin.(72-A)$
$=\sin.(36+A)-(36-A).$ (See Hymer's Trig., p. 48.)

1st. $\text{Sin.}(72+A)=\sin.72\cos.A+\cos.72\sin.A$
$=\cos.18\cos.A+\sin.18\sin.A$

$$=\frac{1}{4}\sqrt{10+2\sqrt{5}}\,.\cos.A+\frac{1}{4}(\sqrt{5}-1)\sin.A$$

(see Article 4);

$$\therefore\sin.(72-A)=\frac{1}{4}\sqrt{10+2\sqrt{5}}\cos.A-\frac{1}{4}(\sqrt{5}-1)\sin.A;$$

$\therefore$ sin. $(72+A)$ − sin. $(72-A) = \frac{1}{2}(\sqrt{5}-1)$ sin. A.

And, sin. A + sin. $(72+A)$—sin. $(72-A) = \frac{\sqrt{5}+1}{2}$ sin. A. (*a*)

2nd. Sin. $(36 + A)$ = sin. 36 cos. A + cos. 36 sin. A

$$= \frac{1}{4}\sqrt{10-2\sqrt{5}}\,.\ \cos. A + \frac{1}{4}(\sqrt{5}+1)\sin. A.$$

(see Art. 4)

and sin. $(36-A) = \frac{1}{4}\sqrt{10-2\sqrt{5}}\cdot$ cos. A $-\frac{1}{4}(\sqrt{5}+1)$ sin. A.

Then, sin. $(36+A)$ − sin. $(36-A) = \frac{1}{2}(\sqrt{5}+1)$ sin. A. (*b*)

From (*a*) and (*b*) we have :—

Sin. A + sin. $(72 + A)$ − sin. $(72 + A)$

= sin. $(36 + A)$ − sin. $(36 - A)$.

(Euler's Formulæ of Verification.)

By the same process the following properties may be proved :—

1. Cos. A + cos. $(72 + A)$ + cos. $(72 - A)$

= cos. $(36 + A)$ + cos. $(36 - A)$.

2. Sin. 3 A = 4 sin. A sin. $(60 + A)$. sin. $(60 - A)$.

3. Cos. 3 A = 4 cos. A cos. $(60 + A)$. cos. $(60 - A)$.

4. Sin. $(72+A)$. sin. $(72-A)$ + sin.^{2} A $= \frac{\sqrt{5}(\sqrt{5}+1)}{8}$.

(7). Multiply 1 and 2 together, thus :—

$$\text{Sin. }(A+B)\text{ sin. }(A-B)=\text{sin.}^2 A \text{ cos.}^2 B - \text{cos.}^2 A \text{ sin.}^2 B.$$
$$= \text{sin.}^2 A - \text{sin.}^2 B.$$
$$= \text{cos.}^2 B - \text{cos.}^2 A.$$

Multiply 3 and 4 together, thus :—

$$\text{Cos. }(A+B)\text{ cos. }(A-B)=\text{cos.}^2 A \text{ cos.}^2 B - \text{sin.}^2 A \text{ sin.}^2 B.$$
$$= \text{cos.}^2 A - \text{sin.}^2 B.$$
$$= \text{cos.}^2 B - \text{sin.}^2 A.$$

EXAMPLES.

1. $\text{Sin. }(A + B)\text{ sin. }(A - B) + \text{sin. }(B + C)\text{ sin. }(B-C) = \text{sin.}^2 A - \text{sin.}^2 C.$
2. $\text{Sin. }(A + B)\text{ sin. }(A - B) + \text{sin. }(B + C)\text{ sin. }(B - C) + \text{sin. }(C + D)\text{ sin. }(C - D) = \text{sin.}^2 A - \text{sin.}^2 D.$
3. $\text{Cos. }(A + B)\text{ cos. }(A - B) - \text{cos. }(B + C)\text{ cos. }(B - C) = \text{cos.}^2 A - \text{cos.}^2 C.$

(D).

On the conversion of (sin. x + sin. y) into the product of two quantities.

Take half the sum of the two angles, thus, $\frac{x+y}{2}$, and half the difference of the two angles, thus $\frac{x-y}{2}$.

$$\text{Then, sin. } x + \text{sin. } y = 2 \text{ sin. } \frac{x+y}{2} \text{ cos. } \frac{x-y}{2} \quad . \quad . \quad (1)$$

This process is of the utmost importance in solving trigonometrical problems . . . . (*a*)

On the conversion of (sin. x cos. y) into the sum of two quantities.

Take the sum of the two angles, thus $x + y$, and the difference of the two angles, thus $x - y$.

$$\text{Then sin. } x \text{ cos. } y = \frac{1}{2}\Big(\text{sin. }(x+y) + \text{sin. }(x-y)\Big) \quad . \quad (2)$$

The remark (*a*) applies with equal force to this process.

An important principle in the application of the above two processes is, that the sine of an angle may be changed for the cosine of an angle, and the cosine for the sine: thus,

$$\text{Sin. A} = \text{cos. } (90 - \text{A}).$$
$$\text{Cos. A} = \text{sin. } (90 - \text{A}).$$

(1). Prove

$\text{Sin.}\, n\,x + \sin.\,(n-2)\,x = 2\cos.\,x\sin.\,(n-1)\,x.$

First, $\dfrac{n\,x+(n-2)x}{2} = \dfrac{(2\,n-2)x}{2} = (n-1)\,x.$

Second, $\dfrac{n\,x-(n-2)x}{2} = \dfrac{2\,x}{2} = x.$

$\therefore \sin.\,n\,x + \sin.\,(n-2)\,x = 2\cos.\,x\sin.\,(n-1)\,x.$

Solve the following examples :—

1. $\text{Sin.}\,2\,x + \sin.\,4\,x = 2\sin.\,3\,x\cos.\,x.$

2. $\text{Sin.}\,x + \sin.\,2\,x = 2\sin.\,\dfrac{3\,x}{2}\cos.\,\dfrac{x}{2}.$

3. $\text{Sin.}\,5\,x + \sin.\,3\,x = 2\sin.\,4\,x\cos.\,x.$

4. $\text{Sin.}\,\dfrac{x+y}{2} + \sin.\dfrac{x-y}{2} = 2\sin.\,\dfrac{x}{2}\cos.\,\dfrac{y}{2}.$

5. $\text{Sin.}\,\dfrac{\pi+x}{2} + \sin.\dfrac{\pi-x}{2} = 2\cos.\,\dfrac{x}{2}.$

6. $\text{Sin.}\,3\,x + \sin.\,x = 2\sin.\,2\,x\cos.\,x.$

7. $\text{Sin.}\,x + \sin.\,5\,x = 2\sin.\,3\,x\cos.\,x.$

8. $\text{Sin.}\,n\,x - \sin.\,(n-2)\,x = 2\sin.\,x\cos.\,(n-1)x.$

The expression

$\text{Sin.}\,n\,x - \sin.\,(n-2)\,x = \sin.\,n\,x + \sin.\,(2-n)\,x.$

Then proceed as above. See (B.)

9. $\text{Sin.}\,3\,x - \sin.\,x = 2\sin.\,x\cos.\,2\,x.$

10. $\text{Sin.}\,5\,x - \sin.\,x = 2\sin.\,2\,x\cos.\,3\,x.$

11. $\text{Sin.}\ (\pi - x) - \sin.\ (\pi + x) = 2 \sin.\ x.$

12. $\text{Sin.}\ (x + y) - \sin.\ (x - y) = 2 \cos.\ x \sin.\ y.$

13. $\text{Sin.}\ x - \sin.\ 5\,x = -2 \sin.\ 2\,x \cos.\ 3\,x.$

14. $\text{Cos}\ \{a + (2n-1)\,x\} + \cos.\ \{a + (2n+1)\,x\} = 2 \cos.\ x \cos.\ (a + 2\,n\,x).$

Observe, that $\cos.\ \{a + (2\,n-1)\,x\} = \sin.\ \{90 - a - (2\,n-1)\,x\}$, &c.

15. $\text{Cos.}\ x + \cos.\ 2\,x = 2 \cos.\ \frac{3\,x}{2} \cos.\ \frac{x}{2}.$

16. $\text{Cos.}\ (x + y) + \cos.\ x = 2 \cos.\ \left(x + \frac{y}{2}\right) \cos.\ \frac{y}{2}$

17. $\text{Cos.}\ 3\,x + \cos.\ x = 2 \cos.\ 2\,x \cos.\ x.$

18. $\text{Cos.}\ 5\,x + \cos.\ 3\,x = 2 \cos.\ 4\,x \cos.\ x.$

19. $\text{Cos.}\ (x + y) + \cos.\ (x - y) = 2 \cos.\ x \cos.\ y.$

20. $\text{Cos.}\ (x + 2\,y) + \cos.\ x = 2 \cos.\ (x + y) \cos.\ y.$

21. $\text{Cos.}\ n\,x + \cos.\ (n - 2)\,x = 2 \cos.\ (n - 1)\,x \cos.\ x.$

22. $\text{Cos.}\ \{a + (2\,n-1)x\} - \cos.\ \{a + (2n+1)x\} = 2 \sin.\ x \sin.\ \{a + 2\,n\,x\}$

Observe, that

$\cos.\ \{a + (2n+1)x\} = \sin.\ \{90 - a - (2n+1)x\} = -\sin\ \{a + (2n+1)x - 90\}$

23. $\text{Cos.}\ 3\,x - \cos.\ x = -2 \sin.\ 2\,x \sin.\ x.$

24. $\text{Cos.}\ (2\,x + y) - \cos.\ (x - 2\,y) = 2 \sin.\ \frac{y - 3x}{2} \sin.\ \frac{x + 3y}{2}.$

25. $\text{Cos.}\ x - \cos.\ 5\,x = 2 \sin.\ 3\,x \sin.\ 2\,x.$

26. $\text{Cos.}\ x - \cos.\ (x + 2\,y) = 2 \sin.\ (x + y) \sin.\ y.$

27. $\text{Cos.}\ \frac{2\,x}{3} - \cos.\ \frac{4\,x}{3} = 2 \sin.\ x \sin.\ \frac{x}{3}.$

28. $\text{Cos. } y - \cos. x = 2 \sin. \frac{x+y}{2} \sin. \frac{x-y}{2}$.

29. $\text{Cos. } (x+y-z) - \cos. (x+y+z) = 2 \sin. (x+y) \sin. z$.

30. $\text{Cos. } (n-1)\, x - \cos. n\,x = 2 \sin. \left(n - \frac{1}{2}\right) x \sin. \frac{x}{2}$.

31. $\text{Cos. } (n\,x - m\,y) - \cos. (n\,x + m\,y) = 2 \sin. n\,x \sin. m\,y$.

32. $\text{Cos. } x - \cos. 7\,x = 2 \sin. 4\,x \sin. 3\,x$.

(2). The object here is to change the product of two trigonometrical functions into the sum or difference of two trigonometrical functions.

1. Prove $\sin. 3\,x \cos. x = \frac{1}{2} (\sin. 4\,x + \sin. 2\,x)$.

Add the angles, thus, $3\,x + x = 4\,x$.

Subtract the angles, thus, $3\,x - x = 2\,x$.

$\therefore \text{Sin. } 3\,x \cos. x = \frac{1}{2} (\sin. 4\,x + \sin. 2\,x)$.

2. $\text{Sin. } 5\,x \cos. 3\,x = \frac{1}{2} (\sin. 8\,x + \sin. 2\,x)$.

3. $\text{Sin. } x \cos. y = \frac{1}{2} \{\sin. (x+y) + \sin. (x-y)\}$

4. $\text{Sin. } \frac{x+y}{2} \cos. \frac{x-y}{2} = \frac{1}{2} (\sin. x + \sin. y)$.

5. $\text{Sin. } (x+y) \cos. x - \sin. x \cos. (x+y) = \sin. y$.

(Jeane's Trig., p. 46.)

6. $\text{Sin. } n\,x \cos. (n-1)x - \sin. \frac{3x}{2} \cos. \frac{x}{2} = \frac{1}{2} \{\sin. (2n-1)x - \sin. 2\,x.\}$

7. $\text{Sin. } 5\,x \sin. x = \frac{1}{2} (\cos. 4\,x - \cos. 6\,x).$

Observe, that $\sin. x = \cos. (90 - x)$.

8. $\text{Sin. } 2\,x \sin. x = \frac{1}{2} (\cos. x - \cos. 3\,x.)$

9. $\text{Sin. } \frac{3\,x}{2} \sin. \frac{x}{2} = \frac{1}{2} (\cos. x - \cos. 2\,x).$

10. $\text{Sin. } n\,x \sin. (n-2)\,x = \frac{1}{2} \{\cos. 2\,x - \cos. (2\,n-2)\,x.\}$

11. $\text{Sin. } x \sin. y = \frac{1}{2} \{\cos. (x-y) - \cos. (x+y).\}$

12. $\text{Sin. } x \sin. (2\,x+y) = \frac{1}{2} \{\cos. (x+y) - \cos. (3\,x+y).\}$

13. $\text{Cos. } 3\,x \cos. x = \frac{1}{2} (\cos. 2\,x + \cos. 4\,y).$

Observe, that $\cos. 3\,x = \sin. (90-3\,x)$.

14. $\text{Cos. } 5\,x \cos. x = \frac{1}{2} (\cos. 6x + \cos. 4\,y).$

15. $\text{Cos. } n\,x \cos. (n-2)x = \frac{1}{2} \{\cos. (2\,n-2)\,x + \cos. 2\,x.\}$

16. $\text{Cos. } x \cos. y = \frac{1}{2} \{\cos. (x+y) + \cos. (x-y).\}$

(3). Combining the above two processes by addition, subtraction, multiplication, division,

&c., as indicated by the problem, the following properties will present but little difficulty.

1. Prove, $\sin.\frac{2x}{3} + \sin.\frac{4x}{3} = \tan. x\left(\cos.\frac{2x}{3} + \cos.\frac{4x}{3}\right)$

(Colenso's Trig.)

Since $\sin.\frac{2x}{3} + \sin.\frac{4x}{3} = 2 \sin. x \cos.\frac{x}{3}$.

And $\cos.\frac{2x}{3} + \cos.\frac{4x}{3} = 2 \cos. x \cos.\frac{x}{3}$.

Divide, and the property is obvious.

2. $\text{Sin}. x + \sin. 3x = \tan. 2x (\cos. x + \cos. 3x)$.

3. $\text{Sin}. x + \sin. y = \tan.\frac{x+y}{2} (\cos. x + \cos. y)$.

4. $\text{Sin}. x + \sin. 5x = \tan. 3x (\cos. x + \cos. 5x)$.

5. $\text{Sin}. x + \sin. 3x + \sin. 5x = \tan. 3x (\cos. x + \cos. 3x + \cos. 5x)$

Add $\sin. 3x$ to $\sin. x + \sin. 5x = 2 \sin. 3x \cos. 2x$, &c.

6. $\text{Sin}. x + \sin. 4x + \sin. 7x = \tan. 4x (\cos. x + \cos. 4x + \cos. 7x)$.

7. $(\text{Sin}. x + \sin. 5x)(\cos. x + \cos. 5x) = \sin. 6x (\cos. 4x + 1)$.

8. $(\text{Sin}. x + \sin. 7x)(\cos. x + \cos. 7x) = \sin. 8x (\cos. 6x + 1)$.

9. $\text{Sin}. (x+y) + \sin. (x-y) = \tan. x \{\cos. (x+y) + \cos. (x-y)\}$

10. $\text{Sin}. x - \sin. 3x = \cot. x (\cos. 3x - \cos. x)$.

11. $\text{Sin}. x - \sin. y = \cot.\frac{x+y}{2} (\cos. y - \cos. x)$.

12. $\text{Sin.}5x + a\cos.3x - \sin.x = \cot.3x(\cos.x + a\sin.3x - \cos.5x)$,

13. $\text{Sin.}7x + a\cos.4x - \sin.x = \cot.4x(\cos.x + a\sin.4x - \cos.7x)$;

14. $(\text{Sin.}\,5x - \sin.\,x)\,(\cos.5\,x - \cos.\,x) = \sin.\,6x\,(\cos.\,4\,x - 1)$.

15. $(\text{Sin.}\,7x - \sin.x)\,(\cos.7x - \cos.x) = \sin.\,8x\,(\cos.\,6\,x - 1)$.

16. $\text{Sin.}(x+y) - \sin.(x-y) = \cot.x\left\{\cos.(x-y) - \cos.(x+y)\right\}$

17. $\text{Vers.}\left(\dfrac{m\,\pi}{m+n} - x\right) + \text{vers.}\left(\dfrac{n\,\pi}{m+n} + x\right) = 2.$

(Jeane's Trig., p. 47.)

$$= 2 - \left\{\cos.\left(\frac{m\,\pi}{m+n} - x\right) + \cos.\left(\frac{n\,\pi}{m+n} + x\right)\right\}.$$

$$= 2 - 2\cos.\frac{\pi}{2}\cos.\left(\frac{m-n}{m-n}\cdot\frac{\pi}{2} - x\right).$$

$$= 2;\ \text{Since, } \cos.\frac{\pi}{2} = 0.$$

18. $2\,\text{vers.}\,\dfrac{\pi + x}{2}\,.\,\text{vers.}\,\dfrac{\pi - x}{2} = \text{vers.}\,(\pi - x.)$

(Jeane's Trig., p. 47.)

19. $\text{Vers.}\left(\dfrac{m\,\pi}{m+n} - x\right) - \text{vers.}\left(\dfrac{n\,\pi}{m+n} + x\right) = 2\sin.\left(\dfrac{m-n}{m+n}\cdot\dfrac{\pi}{2} + x.\right)$

20. $\text{Cos.}^2\,(x + y) - \cos.\,(2\,x + y)\,\cos.\,y = \sin.^2\,x.$

(Jeane's Trig., p. 46.)

$$= \frac{1 + \cos.\,2\,(x+y)}{2} - \frac{\cos.\,(2\,x + 2\,y) + \cos.\,2\,x}{2}$$

$$= \frac{1 - \cos.\,2\,x}{2} = \sin.^2\,x.$$

21. $\text{Sin.}^2\,(x + y) + \cos.\,(2\,x + y)\,\cos.\,y = \cos.^2\,x.$

22. $\text{Cos.}^2 (x + y) + \sin. (2x + y) \sin. y = \cos.^2 x$.

23. $\text{Sin.}^2 (x + y) - \sin. (2x + y) \sin. y = \sin.^2 x$.

24. $\text{Sin.} (x+y) \sin. (x-y) = \frac{1}{2} (\cos. 2y - \cos. 2x)$.

$$\left.\begin{array}{l} = \cos.^2 y - \cos.^2 x \\ = \sin.^2 x - \sin.^2 y. \end{array}\right\} \begin{array}{c}\text{(Jeane's Trig.,} \\ \text{p. 43.)}\end{array}$$

25. $\text{Sin.} (x+y) \sin. (x-y) = \cos.^2 x \cos.^2 y (\tan.^2 x - \tan.^2 y)$.
(Jeane's Trig., p. 47).

26. $\text{Sin.} (x+y) \sin. (x-y) = \sin.^2 x \sin.^2 y (\cot.^2 y - \cot.^2 x)$.

27. $\text{Cos.}(x+y) \cos.(x-y) = \frac{1}{2} (\cos. 2x + \cos. 2y)$.

$$\left.\begin{array}{l} = \cos.^2 x - \sin.^2 y \\ = \cos.^2 y - \sin.^2 x \end{array}\right\} \begin{array}{c}\text{(Jeane's Trig.,} \\ \text{p. 43.)}\end{array}$$

28. $\text{Cos.}(x+y) \cos.(x-y) = \sin.^2 x \cos.^2 y (\cot.^2 x - \tan.^2 y)$.

29. $\text{Sec.}(45+x) \sec.(45-x) = 2 \sec. 2x$. (Jeane's Trig., p. 46.)

$$= \frac{2}{2 \cos. (45+x) \cos. (45-x)}.$$

$$= \frac{2}{\cos. 90 + \cos. 2x}.$$

$$= \frac{2}{\cos. 2x} = 2 \sec. 2x.$$

30. $\text{Sin.}(x+y) \sin.(y+z) = \sin. x \sin. z + \sin. y \sin (x+y+z)$.

Convert these products into sums, or differences, and add :

31. $\text{Cos.} (x+y) \sin. (x-y) + \cos. (y+z) \sin. (y-z)$
$+ \cos. (z+v) \sin. (z-v) + \cos. (v+x) \sin. (v-x) = 0$.

(4). Miscellaneous examples depending upon the combination of the processes explained and illustrated in the preceding three articles :—

1. Prove, $\sin.(x+y) = \sin. x \sin. y(\cot. x + \cot. y)$.

(Jeane's Trig., p. 43.)

Since, $\sin.(x+y) = \sin. x \cos. y + \cos. x \sin. y$.

$$= \sin. x \sin. y\left(\frac{\sin. x \cos. y + \cos. x \sin. y}{\sin. x \sin. y}\right).$$

$$= \sin. x \sin. y(\cot. x + \cot. y).$$

2. $\text{Sin.}(x - y) = \sin. x \sin. y(\cot. y - \cot. x)$.

3. $\text{Sin.}(x+y) = \cos. x \cos. y(\tan. x + \tan. y)$.

4. $\text{Sin.}(x-y) = \cos. x \cos. y(\tan. x - \tan. y)$.

5. $\dfrac{\sin.(x+y)}{\sin. x \sin. y} + \dfrac{\sin.(x-y)}{\cos. x \cos. y} = 2 \operatorname{cosec.} 2x + 2 \cot. 2y$.

6. $\dfrac{\sin.(x-y)}{\sin. x \sin. y} + \dfrac{\sin.(y-z)}{\sin. y \sin. z} + \dfrac{\sin.(z-x)}{\sin. z \sin. x} = 0$.

(Jeane's Trig., p. 47.)

7. $\dfrac{\sin.(x+y)}{\sin. x \sin. y.} - \dfrac{\sin.(y+z)}{\sin. y \sin. z} + \dfrac{\sin.(x-z)}{\sin. x \sin. z} = 0$.

8. $\dfrac{\sin.^2(x+y)\sin.^2(x-y)}{\sin.^2 x \sin.^2 y} + \dfrac{\sin.^2(y+z)\sin.^2(y-z)}{\sin.^2 y \sin.^2 z} + \dfrac{\sin.^2(z+x)\sin.^2(z-x)}{\sin.^2 z \sin.^2 x} = 0$.

9. $\dfrac{\cos.(x+y)}{\sin. x \cos. y} + \dfrac{\cos.(x-y)}{\cos. x \sin. y} = 2 \operatorname{cosec.} 2x + 2 \cot. 2y$.

(5). Prove

$$\sin. x + \sin. y + \sin. z = 4 \sin. \frac{x+y}{2} \sin. \frac{x+z}{2} \sin. \frac{y+z}{2}$$

$$+ \sin. (x+y+z).$$ (From Wrigley.)

$$4 \sin. \frac{x+y}{2} \sin. \frac{x+z}{2} \sin. \frac{y+z}{2} = 2 \sin. \frac{x+y}{2} \left(-\cos. \frac{x+y+2z}{2} + \cos. \frac{x-y}{2} \right)$$

$$= 2 \sin. \frac{-x-y}{2} \cos. \frac{x+y+2z}{2} + 2 \sin. \frac{x+y}{2} \cos. \frac{x-y}{2}$$

$$= \sin. z - \sin. (x+y+z) + \sin. x + \sin. y.$$

Wrigley and Platt have solved this question by resolving the sums of the sines into products. The above method is by resolving products into sums.

1. $4 \cos. \frac{x+y}{2} \cos. \frac{x+z}{2} \cos. \frac{y+z}{2} = \cos. x + \cos. y + \cos. z + \cos. (x+y+z).$

2. $4 \sin. x \sin. y \sin. z = \sin. (y+z-x) - \sin. (x+y+z)$

$+ \sin. (x+y-z) + \sin. (x+z-y).$

3. $4 \cos. x \cos. y \cos. z = \cos. (x+y+z) + \cos. (y+z-x)$

$+ \cos. (x+y-z) + \cos. (x+z-y).$

(From Colenso.)

From these four questions a great variety of properties will readily follow by putting $x+y+z=180$, and $x+y+z=90$, and $x=y=z$, respectively.

(6). Prove —

$$\text{Sin.}\,(x+y+z) = \sin.\,x\cos.\,y\cos.\,z + \cos.\,x\sin.\,y\cos.\,z + \cos.\,x\cos.\,y\sin.\,z - \sin.\,x\sin.\,y\sin.\,z.$$

$$\text{Since } \sin.\,(x+y+z) = \sin.\left\{(x+y)+z\right\}$$

$$= \sin.\,(x+y)\cos.\,z + \cos.\,(x+y)\sin.\,z.$$

Multiply the developments of sin. $(x+y)$ and cos. $(x+y)$, and the property is obvious.

1. $\text{Cos.}\,(x+y+z) = \cos.\,x\cos.\,y\cos.\,z - \sin.\,x\sin.\,y\cos.\,z - \sin.\,x\cos.\,y\sin.\,z - \cos.\,x\sin.\,y\sin.\,z.$

It readily follows from these two properties, that —

2. $\dfrac{\text{Sin.}\,(x+y+z)}{\cos.x\cos.y\cos.z} + \tan.\,x\tan.\,y\tan.\,z = \tan.\,x + \tan.\,y + \tan.\,z.$
(From Wrigley.)

3. $\dfrac{\text{Sin.}\,(x+y+z)}{\sin.x\sin.y\sin.z} + 1 = \cot.x\cot.y + \cot.x\cot.z + \cot.y\cot.z.$

4. $\dfrac{\text{Cos.}\,(x+y+z)}{\sin.x\sin.y\sin.z} + \cot.\,x + \cot.\,y + \cot.\,z = \cot.\,x\cot.\,y\cot.\,z.$
(From Wrigley.)

(7). Prove —

$$\text{Tan.}\,(x+y+z) = \frac{\tan.x + \tan.y + \tan.z - \tan.\,x\tan.y\tan.z}{1 - \tan.x\tan.y - \tan.x\tan.z - \tan.y\tan.z}.$$

(Jeane's Trig., p. 47.)

$$\text{Since } \tan.\,(x+y+z) = \tan.\left\{(x+y)+z\right\}$$

$$= \frac{\tan.\,(x+y) + \tan.\,z}{1 - \tan.\,(x+y)\tan.\,z}.$$

Put in this formula the value of tan. $(x+y)$, and the property is obvious.

From the questions in Articles (6) and (7) various properties may be readily derived, by putting respectively $x+y+z=180, x+y+z=90$, $x=y=z$.

(8). Prove—

$$\text{Tan.}\,(n+1)\,x-\tan.\,n\,x=\frac{\sin.\,x}{\cos.\,n\,x\cos.\,(n+1)\,x}.$$

(From Hymer's Trig.)

$$=\frac{\sin.\,(n+1)\,x}{\cos.\,(n+1)x}-\frac{\sin.\,n\,x}{\cos.\,n\,x}.$$

$$=\frac{\sin.(n+1)x\cos.\,n\,x-\cos.(n+1)x\sin.n\,x}{\cos.\,n\,x\cos.\,(n+1)\,x}$$

$$=\frac{\sin.\,x}{\cos.\,n\,x\cos.\,(n+1)\,x}.$$

1. $\text{Cot.}\,(n+1)\,x-\cot.\,n\,x=-\dfrac{\sin.\,x}{\sin.\,n\,x\sin.\,(n+1)\,x}.$

(9). Prove—

$\text{Tan.}\,(x+y)\,(\sin.\,x\cos.\,x+\sin.\,y\cos.\,y)=\sin.^2x-\sin.^2y.$

$$\text{Since tan.}\,(x+y)=\frac{2\sin.(x+y)\sin.(x-y)}{2\cos.(x+y)\sin.(x-y)}=\frac{\cos.2\,y-\cos.2\,x}{\sin.\,2\,x-\sin.2\,y}$$

$$=\frac{1-2\sin.^2y-1+2\sin.^2x}{\sin.\,2\,x-\sin.\,2\,y}$$

$$=\frac{\sin.^2x-\sin.^2y}{\sin.\,x\cos.\,x-\sin.\,y\cos.\,y}$$

1. $\text{Tan.}\,(x-y)\,(\sin.\,x\cos.\,x+\sin.\,y\cos.\,y)=\sin.^2x-\sin.^2y.$

2. $$\text{Tan.}\,(x+y)=\frac{2\left(\sin.^2\dfrac{x+y+z}{2}-\sin.^2\dfrac{x+y-z}{2}\right)}{\sin.\,(x+y+z)-\sin.\,(x+y-z)}$$

3. $$\frac{1+\cos.x+\cos.2x+\cos.3x}{2\cos.^2x+\cos.x-1}=\frac{2\cos.^2x+2\cos.x\cos.2x}{\cos.2\,x+\cos.\,x}=2\cos.x$$

(10). Prove—

$$1+\tan. x \tan. \frac{x}{2} = \frac{1}{2} \tan. x \left(\tan. \frac{x}{2} + \cot. \frac{x}{2}\right).$$

(Jeane's Trig., p. 47.)

$$\text{Since } 1+\tan. x \tan. \frac{x}{2} = 1 + \frac{2 \tan.^2 \frac{x}{2}}{1-\tan.^2 \frac{x}{2}} = \frac{1+\tan.^2 \frac{x}{2}}{1-\tan.^2 \frac{x}{2}}$$

$$\text{And } \tan. \frac{x}{2} + \cot. \frac{x}{2} = \tan. \frac{x}{2} + \frac{1}{\tan. \frac{x}{2}} = \frac{1+\tan.^2 \frac{x}{2}}{\tan. \frac{x}{2}}$$

$$\therefore \frac{1+\tan. x \tan. \frac{x}{2}}{\tan. \frac{x}{2} + \cot. \frac{x}{2}} = \frac{\tan. \frac{x}{2}}{1-\tan.^2 \frac{x}{2}} = 2 \tan. \frac{x}{2}$$

(11). Prove—

$$(1 + \sec. x) \tan. \frac{x}{2} = \tan. x.$$ (Jeane's Trig., p. 47.)

$$\text{Since } (1 + \sec. x) \tan. \frac{x}{2} = \frac{1+\cos. x}{\cos. x} \times \frac{\sin. \frac{x}{2}}{\cos. \frac{x}{2}}$$

$$= \frac{2 \cos.^2 \frac{x}{2} \sin. \frac{x}{2}}{\cos. x \cos. \frac{x}{2}}$$

The property is obvious.

(12). $1-\sec. x + (1+\sec. x) \tan. \frac{x}{2} = \tan. x \left(1 - \tan. \frac{x}{2}\right)$

(Jeane's Trig., p. 47.)

$$1-\sec. x+(1+\sec. x)\tan.\frac{x}{2}=1+\tan.\frac{x}{2}-\frac{1-\tan.\frac{x}{2}}{\cos. x}$$

$$=\frac{\cos.\frac{x}{2}+\sin.\frac{x}{2}}{\cos.\frac{x}{2}}-\frac{\cos.\frac{x}{2}-\sin.\frac{x}{2}}{\cos.\frac{x}{2}\left(\cos.^2\frac{x}{2}-\sin.^2\frac{x}{2}\right)}$$

$$=\frac{\left(\cos.\frac{x}{2}+\sin.\frac{x}{2}\right)^2-1}{\cos.\frac{x}{2}\left(\cos.\frac{x}{2}+\sin.\right)\frac{x}{2}}=\frac{2\sin.\frac{x}{2}}{\cos.\frac{x}{2}+\sin.\frac{x}{2}}=\frac{2\tan.\frac{x}{2}}{1+\tan.\frac{x}{2}}$$

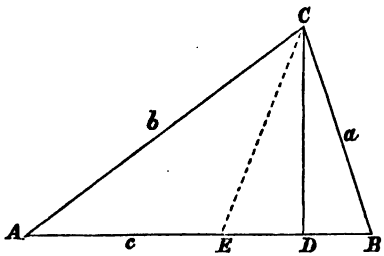
Again, $\tan. x=\dfrac{2\tan.\frac{x}{2}}{1-\tan.^2\frac{x}{2}}$

$$\therefore \tan. x\left(1-\tan.\frac{x}{2}\right)=\frac{2\tan.\frac{x}{2}}{1+\tan.\frac{x}{2}}.$$

Hence, the property is obvious.

(E.)

C D is perp. to B A



$\therefore\ 2c\,.\,AD = b^2 + c^2 - a^2$ and $2c\,.\,BD = a^2 + c^2 - b^2$. 1.

These equations are easily remembered, and as readily proved from the relations—

$$CD^2 = AC^2 - AD^2 = BC^2 - BD^2.$$

$$\text{Cos. } A = \frac{AD}{AC} = \frac{b^2 + c^2 - a^2}{2bc} \quad \ldots\ldots\ldots\ldots \quad 2.$$

$$\text{Cos. } B = \frac{BD}{BC} = \frac{a^2 + c^2 - b^2}{2ac} \quad \ldots\ldots\ldots\ldots \quad 3.$$

$$\text{Cos. } C = = \frac{a^2 + b^2 - c^2}{2ab} \quad \ldots\ldots\ldots\ldots\ldots \quad 4.$$

Since $A + B + C = 180 = \pi$ . . . . 5.

$$\therefore \text{sin. }(A + B) = \text{sin. }C \text{ and cos. }(A + B) = -\cos C \quad 6.$$

$$1 + \text{cos. }A = 1 + \frac{b^2 + c^2 - a^2}{2\,b\,c} = \frac{(a+b+c)(b+c-a)}{2\,b\,c} = 2\text{cos.}^2\frac{A}{2} \quad 7.$$

$$1 - \text{cos. }A = 1 - \frac{b^2 + c^2 - a^2}{2\,b\,c} = \frac{(a+b-c)(a+c-b)}{2\,b\,c} = 2\text{sin.}^2\frac{A}{2} \quad 8.$$

$$\therefore \text{tan.}^2\,\frac{A}{2} = \frac{(a+b-c)\,(a+c-b)}{(a+b+c)\,(b+c-a)} \quad \ldots\ldots \quad 9.$$

Multiply 7 and 8 together.

$$\therefore \text{sin.}^2 A = \frac{4}{b^2 c^2} \cdot \frac{a+b+c}{2} \cdot \frac{a+b-c}{2} \cdot \frac{a+c-b}{2} \cdot \frac{b+c-a}{2} \quad 10.$$

$$\text{Area} = \Delta = \frac{AB\,.\,CD}{2}$$

$$= \frac{b\,c\,\text{sin. }A}{2} = \frac{a\,b\,\text{sin. }C}{2} = \frac{a\,c\,\text{sin. }B}{2} \quad . \quad 11.$$

$$\text{Tan. }\frac{A - B}{2} = \frac{a - b}{a + b}\,\text{cot. }\frac{C}{2} \quad \ldots\ldots\ldots \quad 12.$$

$$\text{Sin. }A + \text{sin. }B + \text{sin. }C = 4\,\text{cos. }\frac{A}{2}\,\text{cos. }\frac{B}{2}\,\text{cos. }\frac{C}{2} \quad 13.$$

$$\text{Cos. }A + \text{cos. }B + \text{cos. }C = 4\,\text{sin. }\frac{A}{2}\,\text{sin. }\frac{B}{2}\,\text{sin. }\frac{C}{2} + 1 \quad 14.$$

In the above formulæ when (A) is substituted for (B), then (a) must be substituted for (b).

(1). If C is a right angle, then, $c^2 = a^2 + b^2$.

$$\text{Hence } A\,D = \frac{b^2}{\sqrt{a^2+b^2}} \text{ and } B\,D = \frac{a^2}{\sqrt{a^2+b^2}} \quad . \quad . \quad . \quad 1.$$

$$A + B = 90\text{, and tan. } \frac{A - B}{2} = \frac{a - b}{a + b} \quad . \quad . \quad . \quad 2.$$

$$\text{Sin. } A = \text{cos. } B = \frac{a}{c}\text{, and sin. } B = \text{cos. } A = \frac{b}{c} \quad 3.$$

From these equations a great variety of properties may be obtained.

EXAMPLES.

1. Cos. $(A - B) =$ cos. A cos. B + sin. A sin. B

$$= \frac{a\,b}{c^2} + \frac{a\,b}{c^2} = \frac{2\,a\,b}{c^2}.$$

2. Sin. $(A-B) = \dfrac{a^2-b^2}{a^2+b^2}$. 3. Tan. $(A-B) = \dfrac{a^2-b^2}{2\,a\,b}$.

4. $\dfrac{\text{versin. }(A-B)}{\text{sin. }(A-B)} = \dfrac{a-b}{a+b}$. 5. $\dfrac{1+\text{cos.}(A-B)}{\text{sin.}(A-B)} = \dfrac{a+b}{a-b}$.

6. Sin. $(45+A) = \dfrac{1}{2}\sqrt{2} \,.\, \dfrac{a+b}{c}$. 7. Cos. $2A = \dfrac{b^2-a^2}{c^2}$.

8. Sin. $2\,A = 2$ sin. A cos. $A = \dfrac{2\,a\,b}{c^2}$.

9. Tan. $2\,A = \dfrac{2\,a\,b}{b^2-a^2}$. 10. Sin.$^2 \dfrac{A}{2} = \dfrac{c-b}{2\,c}$.

11. $\text{Cos.}^2 \frac{A}{2} = \frac{c + b}{2\,c}$. 12. $\text{Tan.}^2 \frac{A}{2} = \frac{c - b}{c + b}$.

13. $\text{Cos.}\,(45 + A) = \frac{1}{2}\sqrt{2}\,\frac{b - a}{c}$.

14. $\text{Tan.}\,(45 - A) = \frac{b - a}{b + a} = -\,\text{tan.}\,\frac{A - B}{2}$.

15. $\text{Sin.}\,(2\,A - B) = \text{sin.}\,2\,A\,\text{cos.}\,B - \text{cos.}\,2\,A\,\text{sin.}\,B$

$$= \frac{b\,(4\,a^2 - c^2)}{c^3}.$$

16. $\text{Cos.}\,(2\,A - B) = \text{sin.}\,3\,A = \frac{a\,(3\,c^2 - 4\,a^2)}{c^3}$.

17. $\text{Cos.}\,3\,A = \frac{b\,(c^2 - 4\,a^2)}{c^3} = \text{sin.}\,(B - 2\,A)$.

18. $\frac{\text{sin.}\,3\,A}{a} - \frac{\text{cos.}\,3\,A}{b} = \frac{2}{c}$.

19. $\Delta = \frac{a\,b}{2} = \frac{c^2}{4} \cdot \frac{2\,a\,b}{c^2} = \frac{c^2}{4}\,\text{sin.}\,2\,A$.

20. $\Delta = \frac{a\,b}{2} = \frac{b^2}{2} \cdot \frac{a}{b} = \frac{b^2}{2}\,\text{tan.}\,A$.

21. $\Delta = \frac{1}{4} \cdot 2\,a\,b = \frac{1}{4}\,(a^2 + b^2 + 2\,a\,b - c^2)$.

$$= \frac{1}{4}\,(a + b + c)\,(a + b - c).$$

22. $\Delta = \frac{a^2\,\text{cot.}\,A}{2} = \frac{1}{2}\left(\frac{a^3 - b^3}{a - b} - c^2\right)$.

Since the lines bisecting the angles of a triangle pass through the centre O of the inscribed circle whose radius is r, we have the triangles

$$\mathrm{AOB} + \mathrm{BOC} + \mathrm{AOC} = \mathrm{ABC}$$

$$\text{or, } r\,a + r\,b + r\,c = a\,b$$

$$\therefore r = \frac{a\,b}{a + b + c}, \text{ and } \mathrm{R} = \frac{c}{2}$$

R being the radius of the circumscribing circle.

23. $r = \dfrac{b}{1 + \cot.\frac{A}{2}} = \dfrac{a}{1 + \cot.\frac{B}{2}} = \dfrac{c}{\cot.\frac{A}{2} + \cot.\frac{B}{2}}$.

24. $r + \mathrm{R} = \dfrac{a + b}{2}$.

25. $r = \dfrac{b\sqrt{c - b}}{\sqrt{c - b} + \sqrt{c + b}}$.

(2). From evident combinations of the formulæ in (E) the following may be readily obtained, and will be true for any triangle whatever.

1. $b\,(b + c - a)$ versin. $\mathrm{A} = a\,(a + c - b)$ versin. B.

2. $(a + b + c)\tan.\dfrac{A}{2}\tan.\dfrac{B}{2} = a + b - c$.

3. $(b + c - a)\tan.\dfrac{A}{2} = (a + c - b)\tan.\dfrac{B}{2}$.

4. $c\,(b + c - a)\sin.^2\dfrac{A}{2} = a\,(a + b - c)\sin.^2\dfrac{C}{2}$.

5. $(a - b)\left(\tan.\dfrac{A}{2} + \tan.\dfrac{B}{2}\right) = c\left(\tan.\dfrac{A}{2} - \tan.\dfrac{B}{2}\right)$.

6. $(b + c - a)\left(\cot.\dfrac{B}{2} + \cot.\dfrac{C}{2}\right) = 2\,a\cot.\dfrac{A}{2}$.

7. $4\,a\cos.\frac{A}{2}\cos.\frac{B}{2}\cos.\frac{C}{2} = (a+b+c)\sin. A.$

8. $(a+b+c)\,\Delta = 2\,a\,b\,c\cos.\frac{A}{2}\cos.\frac{B}{2}\cos.\frac{C}{2}.$

9. $(a+b+c)^3\tan.^2\frac{A}{2}\tan.^2\frac{B}{2}\tan.^2\frac{C}{2} = (a+b-c)(a+c-b)(b+c-a).$

10. $4\,\Delta = (a+b+c)^2\tan.\frac{A}{2}\tan.\frac{B}{2}\tan.\frac{C}{2}.$

11. $8abc\sin.\frac{A}{2}\sin.\frac{B}{2}\sin.\frac{C}{2} = (a+b-c)(a+c-b)(b+c-a).$

(3). From the formulæ 11 in (E) we have :—

$$2\,a\,\Delta = a\,b\,c\sin. A,\text{ and } 2\,b\,\Delta = a\,b\,c\sin. B$$
$$\text{and } 2\,c\,\Delta = a\,b\,c\sin. C.$$

Add these equations, then—

$$2\,(a+b+c)\,\Delta = a\,b\,c\,(\sin. A + \sin. B + \sin. C)$$
$$= 4\,a\,b\,c\cos.\frac{A}{2}\cos.\frac{B}{2}\cos.\frac{C}{2}.$$

EXAMPLES.

1. $2\,\Delta = b\,c\sin. A = b^2.\frac{c}{b}\sin. A = b^2\frac{\sin. C\sin. A}{\sin. B}.$

2. $2\,\Delta\,(\sin. A + \sin. B + \sin. C)$
$= a^2\sin. B\sin. C + b^2\sin. A\sin. C + c^2\sin. A\sin. B.$

3. $8\,\Delta^3 = a^2\,b^2\,c^2\sin. A\sin. B\sin. C.$

4. $2\Delta(\sin.^2 A + \sin.^2 B + \sin.^2 C) = (a^2+b^2+c^2)\sin. A\sin. B\sin. C.$

(4). Since—

$$\text{Tan. A} = \frac{\sin. A}{\cos. A} = \frac{2\,b\,c \sin. A}{2\,b\,c \cos. A} = \frac{4\,\triangle}{b^2 + c^2 - a^2}$$

EXAMPLES.

1. $\text{Tan. B}\,(a^2 + c^2 - b^2) = \tan. C\,(a^2 + b^2 - c^2)$.
2. $4\,\triangle \cot. A = b^2 + c^2 - a^2$.
3. $4\,\triangle\,(\cot. A + \cot. B + \cot. C) = a^2 + b^2 + c^2$.
4. $4\,\triangle \tan. \dfrac{B + C - A}{2} = b^2 + c^2 - a^2$.

Observe that, $\dfrac{B + C - A}{2} = 90 - A$.

(5). Since—

$$\frac{\text{Sin. }(A - B)}{\sin. C} = \frac{\sin. A \cos. B - \cos. A \sin. B}{\sin. C}$$

$$= \frac{2\,a\,c \cos. B - 2\,b\,c \cos. A}{2\,c^2}$$

$$= \frac{a^2 - b^2}{c^2}.$$

EXAMPLES.

1. $2 \sin. (A - B)\,\triangle = (a^2 - b^2) \sin. A \sin. B$.
2. $c(a+b+c)\sin.(A-B) = (a^2-b^2)(\sin.A + \sin.B + \sin.C)$.
3. $c \sin. (A-B) + b \sin. (C - A) + a \sin. (B - C) = 0$.

(6). Since—

$$A\,D + B\,D = c = a \cos. B + b \cos. A \quad . \quad . \quad 1.$$

Similarly, $\quad a = b \cos. C + c \cos. B \quad . \quad . \quad 2.$

Do. $\quad b = c \cos. A + a \cos. C \quad . \quad . \quad 3.$

From these equations the following properties are readily derived:—

EXAMPLES.

1. $(a - b \cos. C) \tan. B = b \sin. C.$ (Jeane's Trig., p. 60.)

2. $(c - b \cos. A) \tan. B = b \sin. A.$

3. $a\,(a - b \cos. C) = c\,(c - b \cos. A).$

Multiply 1, 2, 3 by c, a, b, and add then

4. $a^2 + b^2 + c^2 = 2\,(b\,c \cos. A + a\,c \cos. B + a\,b \cos. C).$

Divide 1 by c, and substitute for $\frac{a}{c}$, $\frac{b}{c}$, then

5. $\text{Sin.}\,(A + B) = \sin. C = \sin. A \cos. B + \cos. A \sin. B.$

From 3 and 2 find cos. A and cos. B, add, then

6. $c\,(\cos. A + \cos. B) = 2\,(a + b) \sin.^2 \frac{C}{2}.$

7. $c\,(\cos. A - \cos. B) = 2\,(b - a) \cos.^2 \frac{C}{2}.$

Multiply 6 and 7 together, then

8. $c^2 (\cos.^2 A - \cos.^2 B) = (b^2 - a^2) \sin.^2 (A + B)$.

Observing, that $\cos. A + \cos. B = 2 \cos.\frac{A+B}{2} \cos.\frac{A-B}{2}$

And $\cos. B - \cos. A = 2 \sin.\frac{A+B}{2} \sin.\frac{A-B}{2}$;

Then, the properties in 6 and 7 become

9. $\text{Cos.}\frac{A-B}{2} = \frac{a+b}{c} \sin.\frac{C}{2}$, and $\sin.\frac{A-B}{2} = \frac{a-b}{c} \cos.\frac{C}{2}$.

(7). In the triangle (E) C E bisects the angle C.

Euclid, 6.3. $a . AE = b . BE$, and $AE + BE = c$.

$\therefore AE = \frac{bc}{a+b}$, and $BE = \frac{ac}{a+b}$ . . . . . 1.

$\angle AEC = B + \frac{C}{2} = 90 - \frac{A-B}{2}$ . . . . . 2.

$\therefore \cot. AEC = \tan.\frac{A-B}{2} = \frac{a-b}{a+b} \cot.\frac{C}{2}$ . . . 3.

EXAMPLES.

1. By triangle and 1, $CE (a + b) = 2 a b \cos.\frac{C}{2}$.

2. The angle, $ECD = \frac{A-B}{2}$ (See Fig., p. 70.)

3. $A\,E' = \dfrac{b\,c}{b-a}$, $B\,E' = \dfrac{a\,c}{b-a}$, $A\,E'\,C = \dfrac{B-A}{2}$.

where C E′ is the external bisector.

4. $D\,E = \dfrac{a-b}{2}\left(\dfrac{c}{a+b} - \dfrac{+b}{c}\right)$

5. From A E C we have, $\cos.\dfrac{A-B}{2} = \dfrac{a+b}{c}\sin.\dfrac{C}{2}$.

6. From 5 and 12 (E), $\sin.\dfrac{A-B}{2} = \dfrac{a-b}{c}\cos.\dfrac{C}{2}$.

7. $(b^2-a^2)\,E\,E' = 2\,a\,b\,c$.

8. Deduce 6 from the triangle A E′ C.

9. From A E′ C, $C\,E' = \dfrac{2\,a\,b}{b-a}\sin.\dfrac{C}{2}$.

10. $D\,E' = \dfrac{a+b}{2}\left(\dfrac{a-b}{c} - \dfrac{c}{a-b}\right)$.

11. $2\,D\,H = b\cos.A - a\cos.B$

$$= c\left\{\frac{\tan.B - \tan.A}{\tan.B + \tan.A}\right\}$$

H is the middle point of A B.

12. Let A F and B G be the bisectors of A and B.

$$\therefore\ B\,F = \frac{a\,c}{b+c},\ \text{and}\ C\,F = \frac{b\,a}{b+c}$$

$$\therefore\ \frac{A\,F\,B}{\Delta} = \frac{B\,F}{a} = \frac{c}{b+c}.$$

$$\text{And, } \frac{\text{E F B}}{\text{A F B}} = \frac{\text{B E}}{c} = \frac{a}{a+b}$$

$$\therefore \text{E F B} = \Delta \frac{a\,c}{(a+b)\,(b+c)}$$

In a similar way the triangles C G F and A G E may be found.

$$13.\quad \Delta\ \text{E F G} = \Delta \frac{2\,a\,b\,c}{(a+b)\,(a+c)\,(b+c)}$$

(8). It is readily seen that if r be the radius of the inscribed circle, whose centre is O, we shall have—

$$(a+b+c)\,r = 2\Delta = b\,c\,\sin.\,A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 1.$$

$$r\left(\cot.\frac{A}{2} + \cot.\frac{B}{2}\right) = c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 2.$$

$$\therefore \frac{r}{c} = \frac{\sin.\frac{A}{2}\ \sin.\frac{B}{2}}{\sin.\ \frac{A+B}{2}} = \frac{2\ \sin.\frac{A}{2}\ \sin.\frac{B}{2}\ \sin.\frac{C}{2}}{\sin.\ C}$$

$$\therefore r\ \sin.\ C = 2\,c\ \sin.\frac{A}{2}\ \sin.\frac{B}{2}\ \sin.\frac{C}{2} \quad . \quad 3.$$

EXAMPLES.

1. $r \sin. A = 2\, a \sin.\frac{A}{2} \sin.\frac{B}{2} \sin.\frac{C}{2}.$

2. $r \sin. B = 2\, b \sin.\frac{A}{2} \sin.\frac{B}{2} \sin.\frac{C}{2}.$

From the above, and 13 in (E), it follows—

3. $2\, r = (a+b+c) \tan.\frac{A}{2} \tan.\frac{B}{2} \tan.\frac{C}{2}.$

4. $2\, r \left(\cot.\frac{A}{2} + \cot.\frac{B}{2} + \cot.\frac{C}{2}\right) = a + b + c.$

5. $2\, r \cot.\frac{B}{2} = a + c - b.$

6. $2\, r \cot.\frac{A}{2} = b+c-a$, and $2\, r \cot.\frac{C}{2} = a+b-c$.

7. $r^2 \cot.\frac{A}{2} \cot.\frac{B}{2} = a\, b \sin^2\frac{C}{2}.$

8. $r^2\left(\cot.^2\frac{A}{2} - \cot.^2\frac{B}{2}\right) = c\ (b-a).$

9. $4\, \Delta = (a + b + c)^2 \tan.\frac{A}{2} \tan.\frac{B}{2} \tan.\frac{C}{2}.$

10. $A\,O \sin.\frac{A}{2} = r \therefore (a+b+c)\, A\,O = 2\, b\, c \cos.\frac{A}{2}.$

11. $\Delta = r^2 \cot.\frac{A}{2} \cot.\frac{B}{2} \cot.\frac{C}{2}.$

(9). Let p, p_1, p_2 represent the perpendiculars from the angles C, A, B upon the sides c, a, b respectively.

R = radius of circumscribing circle.

By Euclid VI. C we have—

$$2 \text{R} p = a\, b;\ 2 \text{R} p_1 = b\, c;\ 2 \text{R} p_2 = a\, c \quad . \quad . \quad 1.$$

$$\left.\begin{aligned} \text{And since } p &= b \sin. \text{A} = a \sin. \text{B} \\ p_1 &= c \sin. \text{B} = b \sin. \text{C} \\ p_2 &= c \sin. \text{A} = a \sin. \text{C} \end{aligned}\right\} \quad . \quad . \quad . \quad 2.$$

$$\therefore 2 \text{R} \sin. \text{A} = a;\ 2 \text{R} \sin. \text{B} = b;\ 2 \text{R} \sin. \text{C} = c. \quad 3.$$

From a combination of these equations the following properties may be readily derived :—

1. $2 \text{R} (\sin. \text{A} + \sin. \text{B} + \sin. \text{C}) = a + b + c.$

2. $8 \text{R} \cos. \frac{\text{A}}{2} \cos. \frac{\text{B}}{2} \cos. \frac{\text{C}}{2} = a + b + c.$

3. $8 \text{R}^3 \sin. \text{A} \sin. \text{B} \sin. \text{C} = a\, b\, c.$

4. $\text{R} \sin. 2\, \text{A} = a \cos. \text{A}.$

5. $a \cos.\text{A} + b \cos.\text{B} + c \cos.\text{C} = \text{R}(\sin.2\text{A} + \sin.2\text{B} + \sin.2\text{C}).$

$$= 4\, \text{R} \sin. \text{A} \sin. \text{B} \sin. \text{C}.$$

6. $b\,p = b^2 \sin.\text{A};\ a\,p = a^2 \sin.\text{B} \therefore (a+b)p = a^2 \sin.\text{B} + b^2 \sin.\text{A}.$

7. $p\,(n + m) = n\, b \sin. \text{A} + m\, a \sin. \text{B}.$

8. $r = 4\,R \sin.\frac{A}{2} \sin.\frac{B}{2} \sin.\frac{C}{2}$.

9. $4\,R\,\Delta = a\,b\,c \;\therefore 2\,r\,R\,(a + b + c) = a\,b\,c$.

10. $(a + b + c)\,\Delta = 2\,a\,b\,c \cos.\frac{A}{2} \cos.\frac{B}{2} \cos.\frac{C}{2}$.

11. $\frac{1}{p} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$.

(10). Draw C O, B O, bisecting the angles C, B, and C O_1, B O_1 bisecting the external angles at C, B.

Since the perpendiculars from O on the sides of the triangle are equal, and the perpendiculars from O_1 on the sides of the triangle produced are also equal, then it follows that the line O O_1 passes through A, and bisects the angle A.

Put r = each perp. from O = radius of inscribed circle.

$r_1 =$ do. $O_1 =$ radius of escribed circle.

$r_2 =$ do. $O_2 =$ radius of escribed circle. (Opposite B.)

$r_3 =$ do. $O_3 =$ radius of escribed circle. (Opposite C.)

The angle O C O_1 = right angle = O B O_1.

Hence the points C, B, O, O_1, are in the circumference of a circle.

Since the triangles

$$A\,O\,B + B\,O\,C + C\,O\,A = A\,B\,C$$

$$A\,O_1\,B + A\,O_1\,C - B\,O_1\,C = A\,B\,C$$

&c. &c. &c.,

It follows that—

$$2\Delta = r(a+b+c) = r_1(b+c-a) = r_2(a+c-b) = r_3(a+b-c) \quad 1.$$

$$\therefore \frac{r}{r_1} = \frac{b+c-a}{a+b+c};\ \frac{r}{r_2} = \frac{a+c-b}{a+b+c};\ \frac{r}{r_3} = \frac{a+b-c}{a+b+c} \quad 2.$$

The following examples are readily derived from equations 1 and 2. Example 8 is obtained in a similar manner to 3 in Art. 8.

1. $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, and $\Delta^2 = r\,r_1 r_2 r_3$.

2. $2\,r_1 r_2 r_3 = (a+b+c)\,\Delta = 2\,a\,b\,c\cos.\frac{A}{2}\cos.\frac{B}{2}\cos.\frac{C}{2}$.

3. $r_1 r_2 + r_1 r_3 + r_2 r_3 = \frac{\Delta^2}{r^2} = \left(\frac{a+b+c}{2}\right)^2$

4. $r_1\left(\tan.\frac{B}{2} + \tan.\frac{C}{2}\right) = a$.

5. $r_1\left(\cot.\frac{A}{2} - \tan.\frac{C}{2}\right) = b$.

6. $r_1\left(\cot.\frac{A}{2} - \tan.\frac{B}{2}\right) = c$.

7. $\therefore 2\, r_1 \cot. \frac{A}{2} = a + b + c.$

8. And $r_1 \sin. A = 2\, a \sin. \frac{A}{2} \cos. \frac{B}{2} \cos. \frac{C}{2}.$

9. Or $r_1 = 4\, R \sin. \frac{A}{2} \cos. \frac{B}{2} \cos. \frac{C}{4}.$

10. $2\, r_1 \tan. \frac{B}{2} = a + b - c.$

11. $r_2 = 4\, R \sin. \frac{B}{2} \cos. \frac{A}{2} \cos. \frac{C}{2}.$

12. $r_3 = 4\, R \sin. \frac{C}{2} \cos. \frac{A}{2} \cos. \frac{B}{2}.$

The properties in 11 and 12 are derived exactly in the same way as the property in 9.

By adding 9, 11, 12 together, and subtracting 8 in Art. 9, there results :—

13. $r_1 + r_2 + r_3 = 4\, R + r.$

These properties might be increased, by various combinations, indefinitely; but the subject has been pursued far enough for our purpose. To those who may be curious in speculations of this nature, we can recommend with confidence Mr. Weddle's Papers in the "Lady's and Gentleman's Diaries."

(11). If Q be the centre of the circumscribing circle of the triangle A B C, and O the centre of the inscribed circle, then—

$$\therefore Q\,A\,O = \frac{A}{2} - (90 - C) = \frac{C-B}{2}$$

$$A\,O = \frac{r}{\sin.\,\frac{A}{2}};\ \text{and } A\,Q = R$$

$$\therefore O\,Q^2 = A\,O^2 + A\,Q^2 - 2\,A\,O\,.\,A\,Q\cos.\,\frac{C-B}{2}.$$

$$= R^2 + \frac{r^2}{\sin.^2\frac{A}{2}} - \frac{2\,R\,r}{\sin.\,\frac{A}{2}}\cos.\,\frac{C-B}{2}.$$

$$= R^2 - 2\,R\,r\left\{\frac{\cos.\,\frac{C-B}{2}}{\sin.\,\frac{A}{2}} - \frac{r}{2\,R\sin.^2\frac{A}{2}}\right\}$$

$$= R^2 - 2\,R\,r\left\{\frac{\cos.\,\frac{C-B}{2} - 2\sin.\,\frac{B}{2}\sin.\,\frac{C}{2}}{\sin.\,\frac{A}{2}}\right\}$$

$$= R^2 - 2\,R\,r.$$

EXAMPLES.

1. $\angle\, Q\,B\,O = \frac{C-A}{2}$, and $\angle\, Q\,C\,O = \frac{B-A}{2}$.

2. $\angle\, Q\,A\,O = \angle\, Q\,B\,O - \angle\, Q\,C\,O$.

(12). Let (a) be the length of the side of a regular polygon of (n) sides, r, R the inscribed and circumscribed circles respectively.

The angle subtended by $(a) = \frac{2\pi}{n}$ . . . . 1.

$\therefore a = 2R \sin.\frac{\pi}{n} = 2r \tan.\frac{\pi}{n}$ . . . . 2.

Area of polygon $= \frac{n a r}{2} = \frac{n a^2}{4} \cot.\frac{\pi}{n}$. 3.

From these three equations the following properties are readily proved :—

1. Area of polygon $= \frac{n R^2}{2} \sin.\frac{2\pi}{n} = n r^2 \tan.\frac{\pi}{n}$.

2. $r = R \cos.\frac{\pi}{n}$.

3. Circum. circle $\times \cos.^2\frac{\pi}{n} =$ In. circle.

4. $r + R = \frac{a}{2} \cot.\frac{\pi}{2n}$.

5. Each angle of polygon $= \frac{\pi(n-2)}{n}$.

6. Sum of angles of polygon $= \pi(n-2)$.

7. If R' and r' be the corresponding radii for a regular polygon of $(2n)$ sides, and of the same perimeter as the former, then

$$R r' = R'^2, \text{ and } r - R = 2r'.$$

Observe, that $a = 2a'$, gives $R \cos. \frac{\pi}{2n} = R'$, and the properties are easily obtained.

(13). Let F, G, H be the middle points of A B, B C, A C of the triangle A B C.

$$\therefore 2\,CF^2 = 2b^2 + \frac{c^2}{2} - 2bc \cos. A. \quad \text{By (2) in (E).}$$

$$= 2b^2 + \frac{c^2}{2} - b^2 - c^2 + a^2$$

$$= a^2 + b^2 - \frac{c^2}{2} \quad . \quad . \quad . \quad 1.$$

EXAMPLES.

1. $4\,(CF^2 + AG^2 + BH^2) = 3\,(a^2 + b^2 + c^2)$.

2. $FD = \frac{b^2 - a^2}{2c}$.

3. C F, A G, B H, meet in a point K.

4. $3\,FK = CF$; $3\,GK = AG$.

(14). If $x + \frac{1}{x} = 2 \cos. A$, and $y + \frac{1}{y} = 2 \cos. B$ in any triangle.

$$\therefore bx + \frac{a}{y} = c. \quad \text{(Wrigley.)}$$

Solve, with respect to x and y.

$$\therefore x = \cos. A + \sqrt{-1} \sin. A, \text{ and } \frac{1}{y} = \cos. B - \sqrt{-1} \sin. B;$$

$$\therefore bx + \frac{a}{y} = b \cos. A + a \cos. B + \sqrt{-1}\, b \sin. A - \sqrt{-1}\, a \sin. B$$

$$= c.$$

EXAMPLES.

1. $a y + \dfrac{b}{x} = c.$

2. $\dfrac{1}{x} = \cos. A - \sqrt{-1} \sin. A.$

3. $x^2 + \dfrac{1}{x^2} = 2 \cos. 2 A$ and $y^2 + \dfrac{1}{y^2} = 2 \cos. 2 B.$

4. $a x (y^2 - 1) = b y (x^2 - 1).$

5. $x y + \dfrac{1}{x y} = 2 \cos. (A + B).$

(F.)

If $\sin. x = a \quad \therefore \quad x = \sin.^{-1} . a.$

" $\cos. x = a$ " $x = \cos.^{-1} . a.$

" $\tan. x = a$ " $x = \tan.^{-1} . a.$

" $\cot. x = a$ " $x = \cot.^{-1} . a.$

" $\sec. x = a$ " $x = \sec.^{-1} . a.$

" $\text{cosec.}\, x = a$ " $x = \text{cosec.}^{-1} . a.$

This notation is convenient, and it is universal; that is, it applies to any function of x.

If $\log. x = a \quad \therefore \quad x = \log.^{-1} . a.$

" $\varphi\ (x) = a$ " $x = \varphi^{-1} . (a).$

Hence, $\sin.^{-1} . a$ is the angle whose sine is a.

" $\cos.^{-1} . a$ " cosine is a.

&c., &c., &c.

(1). Prove that $\sin.^{-1}\dfrac{x}{a} = \tan.^{-1}\dfrac{x}{\sqrt{a^2 - x^2}}$.

$$\text{Let } v = \sin.^{-1}\frac{x}{a} = \tan.^{-1} y$$

$$\therefore \sin. v = \frac{x}{a} \text{ and } \tan. v = y$$

$$\text{Hence, } \cos. v = \frac{\sqrt{a^2 - x^2}}{a} \text{ and } y = \frac{\sin. v}{\cos. v} = \frac{x}{\sqrt{a^2 - x^2}}.$$

The above question may be put thus: verify $\tan. \sin.^{-1}\dfrac{x}{a} = \dfrac{x}{\sqrt{a^2 + x^2}}$; or find the value of $\tan. \sin.^{-1}\dfrac{x}{a}$.

Find the value of the following expressions:—

1. Tan. $\cos.^{-1}\dfrac{x}{a}$.

2. Tan. $\sec.^{-1}\dfrac{x}{a}$.

3. Tan. $\operatorname{cosec.}^{-1}\dfrac{x}{a}$.

4. Tan. $\cot.^{-1}\dfrac{x}{a}$.

5. Cos. $\sin.^{-1}\dfrac{1 - x}{1 + x}$.

6. Sin. $\tan.^{-1}\dfrac{1 - x}{1 + x}$.

7. Cos. $\tan.^{-1}\dfrac{x}{\sqrt{1 - x^2}}$.

8. Tan. $\cos.^{-1}\dfrac{x}{\sqrt{1 + x^2}}$.

9. Sec. $\cos.^{-1}\sqrt{1 - x^2}$.

10. Tan. $\sin.^{-1}\dfrac{x}{\sqrt{1 + x^2}}$.

11. Tan. $\cos.^{-1}\sqrt{1 - x^2}$.

12. Tan. $\sec.^{-1}\dfrac{x}{\sqrt{1 - x^2}}$.

ANSWERS.

1. $\dfrac{\sqrt{a^2 - x^2}}{x}$.
2. $\dfrac{\sqrt{x^2 - a^2}}{a}$.
3. $\dfrac{a}{\sqrt{x^2 - a^2}}$.
4. $\dfrac{a}{x}$.
5. $\dfrac{2\sqrt{x}}{1 + x}$.
6. $\dfrac{1 - x}{\sqrt{2(1 + x^2)}}$.
7. $\sqrt{1 - x^2}$.
8. $\dfrac{1}{x}$.
9. $\dfrac{1}{\sqrt{1 - x^2}}$.
10. x.
11. $\dfrac{x}{\sqrt{1 - x^2}}$.
12. $\sqrt{\dfrac{2x^2 - 1}{1 - x^2}}$.

---

(2). Find the value of tan. 2 cos.$^{-1}$ x.

Let $v = \cos.^{-1} x$ $\therefore$ $\tan. 2 \cos.^{-1} x = \tan. 2 v$.

But $\cos. v = x$ $\therefore$ $\tan. v = \dfrac{\sqrt{1 - x^2}}{x}$.

Hence, $\tan. 2v = \dfrac{2 \tan. v}{1 - \tan.^2 v} = \dfrac{2x\sqrt{1 - x^2}}{2x^2 - 1} = \tan. 2 \cos.^{-1} x$.

Verify the following :—

1. Cos. 2 sin.$^{-1}$ $x = 1 - 2x^2$.

2. Tan. 2 sec.$^{-1}$ $x = \dfrac{2\sqrt{x^2 - 1}}{2 - x^2}$.

3. Cot. 2 sin.$^{-1}$ $3x = \dfrac{1 - 18x^2}{6x\sqrt{1 - 9x^2}}$.

4. Tan. 2 sin.$^{-1}$ $\dfrac{1 - x}{1 + x} = \dfrac{4\sqrt{x}(1 - x)}{6x - x^2 - 1}$.

5. $\text{Sin. } 2 \tan.^{-1} \sqrt{x} = \dfrac{2\sqrt{x}}{1+x}$.

6. $\text{Cos. } 2 \tan.^{-1} 2\sqrt{x} = \dfrac{1-4x}{1+4x}$.

7. $\text{Tan. } 2 \cot.^{-1} \sqrt{1-x^2} = \dfrac{-2\sqrt{1-x^2}}{x^2}$.

8. $\text{Tan. } 2 \operatorname{cosec.}^{-1} \sqrt{x} = \dfrac{2\sqrt{x-1}}{x-2}$.

9. $\text{Tan. } 2 \cos.^{-1} \dfrac{\sqrt{1-x^2}}{x} = \dfrac{2\sqrt{(1-x^2)(2x^2-1)}}{2-3x^2}$.

10. $\text{Tan. } 2 \cos.^{-1} \dfrac{1-x^2}{1+x^2} = \dfrac{4x-4x^3}{x^4-6x^2+1}$.

11. $\text{Sin. } \dfrac{1}{2} \cos.^{-1} x = \sqrt{\dfrac{1-x}{2}}$.

12. $\text{Tan. } \dfrac{1}{2} \sin.^{-1} \dfrac{1-x}{1+x} = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$.

13. $\text{Tan. } \dfrac{1}{2} \sec.^{-1} \sqrt{x} = \sqrt{\dfrac{\sqrt{x}-1}{\sqrt{x}+1}}$.

14. $\text{Cos. } \dfrac{1}{2} \tan.^{-1} \dfrac{1+x}{1-x} = \sqrt{\dfrac{1}{2} + \dfrac{1-x}{\sqrt{8+8x^2}}}$.

(3). Since, $\tan. (A + B) = \dfrac{\tan. A + \tan. B}{1 - \tan. A \tan. B}$

$$\therefore A + B = \tan.^{-1} \frac{\tan. A + \tan. B}{1 - \tan. A \tan. B}.$$

By putting tan. A $= x$ and tan B $= y$

$$\therefore \tan.^{-1} x + \tan.^{-1} y = \tan.^{-1} \frac{x+y}{1-xy} \quad . \quad . \quad . \quad . \quad . \quad 1.$$

$$\text{Similarly, } \tan.^{-1} x - \tan.^{-1} y = \tan.^{-1} \frac{x-y}{1+xy} \quad . \quad . \quad . \quad 2.$$

Equations 1 and 2 are important and should be remembered.

Prove the following :—

1. $\text{Tan.}^{-1} \frac{1}{3} + \tan.^{-1} \frac{1}{4} = \tan.^{-1} \frac{7}{11}$.

2. $\text{Tan.}^{-1} \frac{1}{5} + \tan.^{-1} \frac{1}{6} = \tan.^{-1} \frac{11}{29}$.

3. $\text{Tan.}^{-1} \frac{1}{3} - \tan.^{-1} \frac{1}{4} = \tan.^{-1} \frac{1}{13}$.

4. $\text{Tan.}^{-1} \frac{1}{5} - \tan.^{-1} \frac{1}{6} = \tan.^{-1} \frac{1}{31}$.

5. $\text{Tan.}^{-1} \frac{a+x}{2} + \tan.^{-1} \frac{x-a}{2} = \tan.^{-1} \frac{4x}{4+a^2-x^2}$.

6. $\text{Tan.}^{-1} \frac{a+x}{2} - \tan.^{-1} \frac{x-a}{2} = \tan.^{-1} \frac{4a}{4+x^2-a^2}$.

7. If $x = \sqrt{\frac{a(1+a^2)}{2+a}} \therefore \tan.^{-1}(x-a) + \tan^{-1}(x+a) = \tan.^{-1} \frac{a}{x}$.

8. If $x^2 = a^2 - 3 \therefore \tan.^{-1}(x-a) - \tan.^{-1}(x+a) = \tan.^{-1} a$.

9. $\text{Tan.}^{-1} \frac{1+x}{1-x} - \tan.^{-1} \frac{1-x}{1+x} = \tan.^{-1} \frac{2x}{1-x^2} = 2 \tan.^{-1} x$.

(4). From Eq. 1, last Article, it is evident that if y be determined from $x + y = 1 - x\,y$;

$$\therefore \tan.^{-1} x + \tan.^{-1} \frac{1-x}{1+x} = \tan.^{-1} 1 = \frac{\pi}{4},$$

whatever may be the value of x.

Prove the following :—

1. $\text{Tan.}^{-1} x - \tan.^{-1} \dfrac{x-1}{x+1} = \dfrac{\pi}{4}$.

2. $\text{Tan.}^{-1} x + \tan.^{-1} \dfrac{1 - x\sqrt{3}}{x + \sqrt{3}} = \dfrac{\pi}{6}$.

3. $\text{Tan.}^{-1} x - \tan.^{-1} \dfrac{x\sqrt{3} - 1}{x + \sqrt{3}} = \dfrac{\pi}{6}$.

4. $\text{Tan.}^{-1} x + \tan.^{-1} \dfrac{\sqrt{3} - x}{1 + \sqrt{3}\,x} = \dfrac{\pi}{3}$.

(5). Since,

$$2 \tan.^{-1} x = \tan.^{-1} x + \tan.^{-1} x = \tan.^{-1} \frac{2x}{1-x^2} \quad . \quad . \quad 1.$$

$$\therefore 3 \tan.^{-1} x = 2 \tan.^{-1} x + \tan. x = \tan.^{-1} \frac{x(3-x^2)}{1-3x^2} \cdot \quad 2.$$

$$4 \tan.^{-1} x = 3 \tan.^{-1} x + \tan. x = \tan.^{-1} . \frac{4x(1-x^2)}{1+x^4-6x^2} \quad 3.$$

&c., &c., &c.

1. In 1, make $2x = 1 - x^2 \;\therefore \tan. \dfrac{\pi}{8} = \sqrt{2} - 1$.

2. In 2, make $3x - x^3 = 1 - 3x^2 \;\therefore \tan \dfrac{\pi}{12} = 2 - \sqrt{3}$.

3. In 3, make $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$;

$$\therefore \tan. \frac{\pi}{16} = (\pm \sqrt{2} - 1) \pm \sqrt{4 \pm 2\sqrt{2}}.$$

(6). Since,

$$2 \tan.^{-1} x + \tan.^{-1} y = \tan.^{-1} \frac{2x}{1-x^2} + \tan.^{-1} y$$

$$= \tan.^{-1} \frac{2x + y(1-x^2)}{1 - x^2 - 2xy}.$$

1. Take, $2x + y(1-x^2) = 1 - x^2 - 2xy$

$$\therefore 2 \tan.^{-1} x + \tan.^{-1} \frac{1 - x^2 - 2x}{1 - x^2 + 2x} = \frac{\pi}{4}.$$

2. If $x = \frac{1}{3}$ in Ex. 1 $\therefore 2 \tan.^{-1} \frac{1}{3} + \tan.^{-1} \frac{1}{7} = \frac{\pi}{4}$.

3. $2 \tan.^{-1} x - \tan.^{-1} y = \tan.^{-1} \frac{2x - y(1-x^2)}{1 - x^2 + 2xy}$.

4. Take, $2x - y(1-x^2) = 1 - x^2 + 2xy$

$$\therefore 2 \tan.^{-1} x - \tan.^{-1} \frac{x^2 + 2x - 1}{-x^2 + 2x + 1} = \frac{\pi}{4}.$$

(7). $3 \tan.^{-1} x + \tan.^{-1} y = \tan.^{-1} \frac{x(3-x^2)}{1-3x^2} + \tan.^{-1} y$

$$= \tan.^{-1} \frac{3x - x^3 + y(1-3x^2)}{1 - 3x^2 - y(3x - x^3)}.$$

1. If $3x - x^3 + y(1 - 3x^2) = 1 - 3x^2 - y(3x - x^3)$

$$\therefore 3 \tan.^{-1} x + \tan.^{-1} \frac{1 - 3x - 3x^2 + x^3}{1 + 3x - 3x^2 - x^3} = \frac{\pi}{4}.$$

(8). $4\tan.^{-1}x + \tan.^{-1}y = \tan.^{-1}\dfrac{4x(1-x^2)}{1+x^4-6x^2} + \tan.^{-1}y$

$$= \tan.^{-1}\frac{4x(1-x^2)+y(1+x^4-6x^2)}{1+x^4-6x^2-4xy(1-x^2)}.$$

1. If $4x(1-x^2)+y(1+x^4-6x^2)=1+x^4-6x^2-4xy(1-x^2)$

$$\therefore\ 4\tan.^{-1}x + \tan.^{-1}\frac{x^4+4x^3-6x^2-4x+1}{x^4-4x^3-6x^2+4x+1} = \frac{\pi}{4}.$$

2. If $x = \dfrac{1}{5}$ in Ex. 1 $\therefore\ 4\tan.^{-1}\dfrac{1}{5} - \tan.^{-1}\dfrac{1}{239} = \dfrac{\pi}{4}$.

3. $4\tan.^{-1}\dfrac{1}{5} - \tan.^{-1}\dfrac{1}{70} + \tan.^{-1}\dfrac{1}{99} = \dfrac{\pi}{4}$.

TRIGONOMETRICAL EQUATIONS.

(1). Given, sin. $(a + b\,x) = \cos.\ (c + d\,x)$, to find x.

$$\text{Sin.}(a+bx)=\sin.\left(\frac{\pi}{2}-c-d\,x\right)=\sin.\left(\frac{\pi}{2}-c-dx\pm 2n\pi\right)$$

$$\therefore a + b\,x = \frac{\pi}{2} - c - d\,x \pm 2\,n\,\pi$$

$$\therefore x = \frac{(1 \pm 4\,n)\,\frac{\pi}{2} - c - a}{b + d}.$$

n is any whole number.

EXAMPLES.

1. Sin. $2\,x = \cos.\ x \therefore x = (1 \pm 4\,n)\,\dfrac{\pi}{6}$.

2. Cosec.$^2 \dfrac{x}{2} - \sec^2 \dfrac{x}{2} = 2\,\sqrt{3}$ cosec.$^2\,x$. Ans. $2\,n\,\pi \pm \dfrac{\pi}{6}$.

3. Cos. $n\,x$ + cos. $(n - 2)\,x$ = cos. x. Ans. $\dfrac{(6m \pm 1)\,\pi}{3\,(n-1)}$.

4. Sin. x — cos. $x = 4\cos.^2 x \sin. x$. Ans. $(\pm 4m - 1)\dfrac{\pi}{4}$.

5. Sin. $(x + a)$ + cos. $(x + a)$ = sin. $(x - a)$ + cos. $(x - a)$.
Ans. 45.

6. Tan. $x + 2\cot. 2x = \sin. x\left(1 + \tan. x \tan. \dfrac{x}{2}\right)$. Ans. $(4n \pm 1)\dfrac{\pi}{4}$.

7. Sin. $7x$ — sin. x = sin. $3x$. Ans. $\dfrac{m\pi}{3}$, or, $\left(\dfrac{6m \pm 1}{12}\right)\pi$.

(2). Given, $a \cos. n\,x + b \sin. n\,x = c$, where $c < \sqrt{a^2 + b^2}$.

$$\therefore \frac{a}{\sqrt{a^2+b^2}} \cos. n\,x + \frac{b}{\sqrt{a^2+b^2}} \sin. n\,x = \frac{c}{\sqrt{a^2+b^2}}$$

$$\text{Put, } \sin. h = \frac{a}{\sqrt{a^2+b^2}} \therefore \cos. h = \frac{b}{\sqrt{a^2+b^2}}.$$

$$\therefore \sin. h \cos. n\,x + \cos. h \sin. n\,x = \sin. i,$$

where $\sin. i = \dfrac{c}{\sqrt{a^2+b^2}}$

$$\therefore \sin. (h + n\,x) = \sin. i = \sin. (i \pm 2m\pi)$$

$$\therefore x = \frac{i - h \pm 2m\pi}{n}.$$

EXAMPLES.

1. $\sqrt{2}\,(\cos.\,3\,x + \sin.\,3\,x) = 1.$ Ans. $\frac{\pi}{36}\,(7 + 24\,r).$

2. $\text{Sin.}\,x - \cos.\,x = 4\cos.^2 x \sin.\,x.$ Ans. $(4\,m + 3)\,\frac{\pi}{8}.$

3. $\text{Sin.}\,x + \sqrt{3}\cos.\,x = \frac{1}{2}\,(\sqrt{5} - 1).$ Ans. $\left(2\,r - \frac{7}{30}\right)\pi.$

4. $\text{Sin.}\,2\,x + \cos.\,2\,x = \sqrt{2}.$ Ans. $\left(\frac{1}{8} + r\right)\pi.$

5. $\sqrt{3}\sin.\,4\,x - \cos.\,4\,x = \frac{1}{2}\,\sqrt{5} - 1).$ Ans. $\left(\frac{1}{15} + \frac{r}{2}\right)\pi.$

6. $\sqrt{3}\sin.\,4\,x - \cos.\,4\,x + \sqrt{2}.$ Ans. $\left(r + \frac{5}{24}\right)\frac{\pi}{2}.$

(3). $\text{Sin.}\,n\,x + \sin.\,m\,x = 2\cos.\,h\sin.\,\frac{m+n}{2}\,x$ can be solved.

$$\text{Sin.}\,n\,x + \sin.\,m\,x = 2\sin.\,\frac{n+m}{2}\,x\cos.\,\frac{n-m}{2}\,x$$

$$\therefore \cos.\,\frac{n-m}{2}\,x = \cos.\,h\,;\ \text{or, } x = \frac{4\,r\,\pi + 2\,h}{n-m}.$$

EXAMPLES.

1. $\text{Sin.}\,n\,x + \sin.\,m\,x = 2\sin.\,h\cos.\,\frac{m-n}{2}\,x.$

2. $\text{Cos.}\,n\,x + \cos.\,m\,x = 2\cos.\,h\cos.\,\frac{n+m}{2}\,x.$

3. $\text{Cos. } n\,x + \cos.\, m\,x = 2\cos.\, h \cos.\, \frac{n-m}{2}\,x.$

4. $\text{Sin. } n\,x - \sin.\, m\,x = 2\sin.\, h \cos.\, \frac{m+n}{2}\,x.$

5. $\text{Tan. } x + \cot.\, x = 4 \therefore x = \frac{\pi}{12}\,(1 + 12\,r).$

6. $\text{Cos. } x + \sin.\, x = \sqrt{2} \therefore \left(2\,m + \frac{1}{4}\right)\pi.$

ANSWERS.

1. $\frac{2\,h + 4\,r\,\pi}{n+m}.$ 2. $\frac{4\,r\,\pi + 2\,h}{n-m}.$

3. $\frac{4\,r\,\pi + 2\,h}{n+m}.$ 4. $\frac{2\,h + 4\,r\,\pi}{n-m}.$

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(4). $\text{Sin. } (a + n\,x) \sin.\, (b + n\,x) = \frac{1}{2}\cos.\, c$; can be solved.

Change the product to equal difference.

$$\therefore \cos.\, (a-b) - \cos.\, (2\,n\,x + a + b) = \cos.\, c$$

$$\therefore x = \frac{\cos.^{-1}\{\cos.\, (a-b) - \cos.\, c\} - a - b}{2\,n}.$$

EXAMPLES.

1. $\text{Sin.}\,(n\,x+a)\,\text{sin.}\,(m\,x+b)=\frac{1}{2}\cos.\,c-\frac{1}{2}\cos.\left\{(n+m)\,x+a+b\right\}.$

2. $\text{Sin.}\,(n\,x+a)\,\text{sin.}\,(m\,x+b)=\frac{1}{2}\cos.\,c+\frac{1}{2}\cos.\left\{(n-m)\,x+a-b\right\}.$

3. $\text{Sin.}\,x\,\text{sin.}\,(2\,a+x)+n\cos.^2 a=0.$

4. $\text{Tan.}\,(x+a)\,\tan.\,(x-a)=1;\ \therefore\ x=\left(\frac{2\,m+1}{4}\right)\pi.$

5. $\text{Tan.}\,x+\cot.\,x=2\,\text{cosec.}\,a\ \therefore\ x=\frac{a}{2}\pm r\,\pi.$

6. $\text{Tan.}^3 x=\tan.(x-a)\ \therefore\ x=\frac{1}{4}\left\{\sin.^{-1}(3\sin.\,a)+a\pm 2\,r\,\pi\right\}$

$$\therefore\ 2\sin.^3 x\cos.\,(x-a)=2\cos.^3 x\sin\,(x-a)$$

$$\text{Sin.}^2 x\left\{\sin.(2x-a)+\sin.a\right\}=\cos.^2 x\left\{\sin.(2x-a)-\sin.a\right\}$$

$$\text{or, }\frac{\sin.\,(2\,x-a)+\sin.\,a}{\sin.\,(2\,x-a)-\sin.\,a}=\frac{\cos.^2 x}{\sin.^2 x}$$

$$\therefore\ \frac{\sin.\,(2\,x-a)}{\sin.\,a}=\frac{\cos.^2 x+\sin.^2 x}{\cos.^2 x-\sin.^2 x}=\frac{1}{\cos.\,2\,x}$$

$$\text{and, }\sin.\,(2\,x-a)\cos.\,2\,x=\sin.\,a.$$

ANSWERS.

1. $\dfrac{2\,r\,\pi\pm c+b-a}{n-m}.$ 2. $\dfrac{(\pm 2\,r+1)\,\pi\pm c-a-b}{n+m}.$

3. $x=\cos.^{-1}\left\{\pm\sqrt{1+n}\cos.\,a\right\}-a.$

---

(5). Since sin. $2x =$ cos. $3x$; gives $x = 15$, or -54, it is not difficult to show, as in Art. (4), p. 49, that—

$$4 \sin.^2 x + 2 \sin. x = 1; \text{ gives } x = 18, \text{ or } -54 \text{ deg.} \quad . \quad 1.$$

$$\text{And } 4 \sin.^2 x - 2 \sin. x + 1; \text{ gives } x = 54, \text{ or } -18 \quad . \quad 2.$$

Multiply 1 and 2 together, then—

$$16 \sin.^4 x - 12 \sin.^2 x + 1 = 0 \text{ gives } x = \pm 18, \text{ or } \pm 54 \quad . \quad . \quad 3.$$

Solve the following:—

1. $2 \cos. 2x = 2 \sin. x + 1$. Ans. 18, or -54.

 Observe, that $2 \cos. 2x = 2 - 4 \sin.^2 x$.

2. $\text{Sin.}^2 2x - \sin.^2 x = \frac{1}{4}$. Ans. ± 18, or ± 54.

3. $4 \sin. x \sin. 3x = 1$. „

4. $\text{Sin.} a + \sin. (x - a) + (\sin. 2x + a) = \sin. (x + a) + \sin. (2x - a)$.

 Ans. 36, or 108.

Since $\sin. a + 2 \cos. 2x \sin. a = 2 \cos. x \sin. a$,

$$\therefore 1 + 2 \cos. 2x = 2 \cos. x,$$

which is the complement of equation 2.

5. $4 \cos. x \cos. 3x + 1 = 0$. Ans. 36, or 72.

(6). Solve $\cos. 3x + \cos. 2x + \cos. x = 0$.

$$\therefore 2 \cos. 2x \cos. x + \cos. 2x = 0.$$

$$\text{Or, } \cos. 2x\,(2 \cos. x + 1) = 0.$$

$$\therefore \cos. 2x = 0 = \cos. \frac{\pi}{2} = \cos. \left(\frac{\pi}{2} \pm 2r\pi\right)$$

$$\therefore x = \left(\frac{1}{4} \pm 2r\right)\pi.$$

$$\text{Again, } \cos. x = -\frac{1}{2} = \cos. \left(\frac{2\pi}{3} \pm 2r\pi\right)$$

$$\therefore x = \left(\frac{2}{3} \pm 2r\right)\pi.$$

EXAMPLES.

1. $\text{Sin. } 3x + \sin. 2x + \sin. x = 0$. Ans. $\left(\frac{2}{3} \pm 2r\right)\pi$.

2. $\text{Sin. } 7x - \cos. 3x + \sin. x = 0$. Ans. $\left(\frac{1}{12} \pm r\right)\frac{\pi}{2}$.

3. $\text{Sin. } 3x + \sin. 2x = \sin. x$. Ans. 60, or 180.

4. $\text{Sin. } (x+a) + \cos. (x+a) = \sin. (x-a) + \cos. (x-a)$.

   Ans. $\left(\frac{1}{4} \pm r\right)\pi$.

5. $\text{Sin. } 7x + \sin. 4x + \sin. x = 0$. Ans. $(6r \pm 2)\frac{\pi}{9}$.

6. $\text{Sin. } 9x + \sin. 5x + 2 \sin.^2 x = 1$. Ans. 45, or $\frac{\pi}{42}$.

7. $\text{Sin. } x + \sin. 2x + \sin. 3x + \sin. 4x = 0$.

   Ans. 90, or 180, or 72.

(7). Sin. $(a+nx)=c$ sin. $(b+nx)$.

By developing, we find, $\tan. nx=\dfrac{\sin. a-c\sin. b}{c\cos. b-\cos. a}$.

Another method is as follows:—

$$\frac{\sin.(a+nx)}{\sin.(b+nx)}=c.$$

$$\therefore \frac{\sin.(a+nx)-\sin.(b+nx)}{\sin.(a+nx)+\sin.(b+nx)}=\frac{c-1}{c+1}=h.$$

Change these into products, then—

$$x=-\frac{a+b}{2n}+\frac{1}{n}\cot.^{-1}\left(h\cot.\frac{a-b}{2}\right).$$

EXAMPLES.

1. $m\tan.(a-x)\cos.^2(a-x)=n\tan. x\cos.^2 x.$

$$\therefore 2x=a-\tan.^{-1}\left(\frac{n-m}{n+m}\tan. a\right).$$

2. Cot. $x=n\cot.(a-x)$ $\therefore 2x=a-\sin.^{-1}\left(\dfrac{n-1}{n+1}\sin. a\right)$.

3. Tan. $x=(2+\sqrt{3})\tan.\dfrac{x}{3}$. Ans. ± 45.

4. Sec. $(x+a)+\sec.(x-a)=2\sec. x$.

Ans. $\cos.^{-1}\left(\sqrt{2}\cos.\dfrac{a}{2}\right)$.

(8). Tan. a tan. x=tan.^{2} $(a+x)$—tan.^{2} $(a-x)$.

$$= \sec.^2 (a+x) - \sec.^2 (a-x).$$

$$= \frac{\cos.^2 (a-x) - \cos.^2 (a+x)}{\cos.^2 (a+x) \cdot \cos.^2 (a-x)}.$$

$$\therefore \frac{\sin. a \ \sin. x}{\cos. a \ \cos. x} = \frac{4 \cos. a \ \cos. x \ \sin. a \ \sin. x}{\cos.^2 (a+x) \ \cos.^2 (a-x)}.$$

or, $2 \cos. (a+x) \cos. (a-x) = \pm 4 \cos. a \cos. x$;

or, $\cos. 2a + \cos. 2x = \pm 4 \cos. a \cos. x$;

or, $\cos.^2 x \mp 2 \cos. a \cos. x + \cos.^2 a = 1$;

$$\therefore \cos. x \mp \cos. a = \pm 1;$$

or, $x = \cos.^{-1} (\pm 1 \pm \cos. a)$.

EXAMPLES.

1. Sin. a sin. x = sin.^{2} $(a+x)$ − sin.^{2} $(a-x)$.

Ans. $\cos.^{-1} \left(\frac{\text{Sec. } a}{4}\right)$.

2. Cot. a cot. x = cot.^{2}$(a+x)$ − cot.^{2} $(a-x)$.

Ans. $\sin.^{-1} (\pm \sqrt{2} \mp 1) \sin. a$.

HIGHER TRIGONOMETRY.

$$\cos. n x \pm \sqrt{-1} \sin. n x = (\cos. x \pm \sqrt{-1} \sin. x)^n \quad 1.$$

$$2 \cos. x = \epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad 2.$$

$$2 \sqrt{-1} \sin. x = \epsilon^{x\sqrt{-1}} - \epsilon^{-x\sqrt{-1}} \quad . \ . \ . \ . \ . \ . \ . \quad 3.$$

From (2) and (3) it readily follows, by adding and subtracting, that—

$$\epsilon^{\pm x\sqrt{-1}} = \cos. x \pm \sqrt{-1} \sin. x \quad . \ . \ . \ . \ . \ . \ . \ . \quad 4.$$

Equations (2) and (3) were first obtained by Euler from the developments of cos. x, sin. x, and ϵ^x, in ascending powers of x; and they were considered by Lagrange as the greatest analytical discoveries of the age. (Le Calcul des Fonctions, p. 114.)

Equation (1) was first given by Demoivre, and is called Demoivre's Theorem. It is regarded by Lagrange and Laplace as of equal importance with that of the Binomial Theorem. (Woodhouse's Trig., p. 55.)

The demonstration of (1) readily follows from equation (4), which is true for any value of x; therefore it is true for $n\,x$;

$$\therefore \cos. n\,x \pm \sqrt{-1}\,\sin. n\,x = \epsilon^{\pm n\,x\sqrt{-1}}$$
$$= (\epsilon^{\pm x\sqrt{-1}})^n$$
$$= (\cos. x \pm \sqrt{-1}\,\sin. x)^n$$

which is Demoivre's theorem.

Equations (2) and (3) are true for any value of x; they are true, therefore, for $n\,x$;

$$\therefore 2\cos. n\,x = \epsilon^{n\,x\sqrt{-1}} + \epsilon^{-n\,x\sqrt{-1}}$$
$$= (\epsilon^{x\sqrt{-1}})^n + (\epsilon^{-x\sqrt{-1}})^n.$$

$$\text{Put, } \epsilon^{x\sqrt{-1}} = y \;\therefore\; \epsilon^{-x\sqrt{-1}} = \frac{1}{y}.$$

$$\text{Hence, } 2\cos. x = y + \frac{1}{y}, \text{ and } 2\cos. n\,x = y^n + \frac{1}{y^n}. \qquad 5.$$

This equation was first given by Demoivre.

In a similar manner we may obtain—

$$\left.\begin{aligned} 2\sqrt{-1}\,\sin. x &= y - \frac{1}{y} \\ \text{and, } 2\sqrt{-1}\,\sin. n\,x &= y^n - \frac{1}{y^n} \end{aligned}\right\} \quad \ldots\ldots \quad 6.$$

(1). From (1) :—

$$\text{Cos. } n\,x \pm \sqrt{-1} \sin. n\,x = (\cos. x \pm \sqrt{-1} \sin. x)^n$$
$$= \cos.^n x\,(1 \pm \sqrt{-1} \tan. x)^n.$$

Expanding this expression by the binomial theorem, and equating impossible and possible quantities on each side of the equation, we have—

$$\text{Cos.} n x = \cos.^n x \left\{ 1 - \frac{n(n-1)}{1.2} \tan.^2 x + \frac{n(n-1)\ldots(n-3)}{1.2\ldots4} \tan.^4 x - \&c. \right\}$$

$$\text{Sin.} n x = \cos.^n x \left\{ n \tan. x - \frac{n(n-1)(n-2)}{1.2.3} \tan.^3 x + \&c. \right\}$$

Putting $n\,x = \theta$, these theorems become—

$$\text{Cos.}\theta = (\cos. x)^{\frac{\theta}{x}} \left\{ 1 - \frac{\theta(\theta-x)}{1.2} \left(\frac{\tan. x}{x}\right)^2 + \frac{\theta(\theta-x)\ldots(\theta-3x)}{1.2\ldots4} \left(\frac{\tan. x}{x}\right)^4 - \&c. \right\}$$

$$\text{Sin.}\,\theta = (\cos. x)^{\frac{\theta}{x}} \left\{ \theta \left(\frac{\tan. x}{x}\right) - \frac{\theta(\theta-x)(\theta-2x)}{1.2.3} \left(\frac{\tan. x}{x}\right)^3 + \&c. \right\}$$

By making various suppositions with respect to x and θ, various theorems will follow, some of which are of historical interest. For instance, make $x = 0 \;\therefore\; (\cos. x)^{\frac{\theta}{x}} = 1$ and $\left(\frac{\tan. x}{x}\right) = 1$

$$\therefore \cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \&c.$$

$$\text{And, } \sin.\,\theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \&c.$$

This method of deducing the values of cos. θ and sin. θ in series was first used by Euler.

(2). Expanding the formula in (5) by the binomial theorem, and making obvious arrangement of the terms, there results—

$$2^{n-1}\cos.^n x = \cos. n x + n \cos.(n-2)x + \frac{n(n-1)}{1.2}\cos.(n-4)x + \&c.$$

This series must continue to cos. x when n is odd, and to cos. 0 when n is even. When n is even the last term must be divided by 2.

(3). Expanding the formula in (6) by the binomial theorem, and arranging the terms as above.

For n even :—

$$2^{n-1}(-1)^{\frac{n}{2}}\sin.^n x = \cos. n x - n\cos.(n-2)x + \frac{n(n-1)}{1.2}\cos.(n-4)x - \&c.$$

This series must continue to cos. 0, and the last term must be divided by 2.

For n odd :—

$$2^{n-1}(-1)^{\frac{n-1}{2}}\sin.^n x = \sin. n x - n\sin.(n-2)x + \frac{n(n-1)}{1.2}\sin.(n-4)x - \&c.$$

This series must continue to sin. x.

In deducing the above theorems it is necessary to observe, in arranging the terms, that the coefficients of any two terms equally distant from the beginning and end are the same.

(4). From equation 4 we obtain, by taking the log. of each side,—

$$x \sqrt{-1} = \log. \cos. x + \log. (1 + \sqrt{-1} \tan. x)$$

$$\text{and, } - x \sqrt{-1} = \log. \cos. x + \log. (1 - \sqrt{-1} \tan. x$$

$$\therefore 2x \sqrt{-1} = \log. (1 + \sqrt{-1} \tan. x) - \log. (1 - \sqrt{-1} \tan. x.$$

Expanding the right-hand side of this equation by means of the logarithmic theorem, there results—

$$x = \tan. x - \frac{1}{3} \tan.^3 x + \frac{1}{5} \tan.^5 x - \&c.$$

a series which is usually called Gregory's, being first deduced by that eminent mathematician. The value of x can only be taken between the limits $\frac{1}{4} \pi$ and $- \frac{1}{4} \pi$.

The above series may be put as follows:—

$$\text{Tan.}^{-1} y = y - \frac{1}{3} y^3 + \frac{1}{5} y^5 - \frac{1}{7} y^7 + \&c.$$

(5). Find the $2n$ roots of $x^{2n} - 1 = 0$, n being any whole number.

$$x^{2n} = 1 = \cos. 2 r \pi \pm \sqrt{-1} \sin. 2 r \pi$$

$$\therefore x = (\cos. 2 r \pi \pm \sqrt{-1} \sin. 2 r \pi)^{\frac{1}{2n}}$$

$$= \cos. \frac{r \pi}{n} \pm \sqrt{-1} \sin. \frac{r \pi}{n}.$$

(By Demoivre's Theorem.)

Take $r = 0 \therefore x - 1 = 0$.

" $r = 1 \therefore x^2 - 2x\cos.\frac{\pi}{n} + 1 = 0$.

" $r = 2 \therefore x^2 - 2x\cos.\frac{2\pi}{n} + 1 = 0$.

" $r = 3 \therefore x^2 - 2x\cos.\frac{3\pi}{n} + 1 = 0$.

$\vdots \quad \vdots$

" $r = n \therefore x + 1 = 0$

$$\therefore x^{2n} - 1 = (x-1)(x+1)\left(x^2 - 2x\cos.\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos.\frac{2\pi}{n} + 1\right)\ldots$$

$$\text{to}\left(x^2 - 2x\cos.\frac{(n-1)\pi}{n} + 1\right).$$

Solve the following examples :—

1. $x^{2n+1} - 1 = (x-1)\left(x^2 - 2x\cos.\frac{2\pi}{2n+1} + 1\right)\ldots\left(x^2 - 2x\cos.\frac{2n\pi}{2n+1} + 1\right)$

2. $x^{2n} + 1 = \left(x^2 - 2x\cos.\frac{\pi}{2n} + 1\right)\ldots\left(x^2 - 2x\cos.\frac{(2n-1)\pi}{2n} + 1\right)$

3. $x^{2n+1} + 1 = (x+1)\left(x^2 - 2x\cos.\frac{\pi}{2n+1} + 1\right)\ldots\left(x^2 - 2x\cos.\frac{(2n-1)\pi}{2n+1} + 1\right)$

(6). By giving particular values to x in the theorems of the last article we may obtain various results, as follows :—

Since, $\frac{x^{2n} - 1}{x - 1} = (x + 1)\left(x^2 - 2x\cos.\frac{\pi}{n} + 1\right)\ldots$

$$\left(x^2 - 2x\cos.\frac{(n-1)\pi}{n} + 1\right)$$

and, $\frac{x^{2n} - 1}{x - 1} = x^{2n-1} + x^{2n-2} + \ldots 1$ by division

$= 2n$, when $x = 1$.

$$\therefore 2n = 2\left(2 - 2\cos.\frac{\pi}{n}\right)\ldots\left(2 - 2\cos.\frac{(n-1)\pi}{n}\right)$$

$$\therefore n = 2^{n-1}\left(1 - \cos.\frac{\pi}{n}\right)\ldots\left(1 - \cos.\frac{(n-1)\pi}{n}\right)$$

$$= 2^{n-1}\,.\,2\sin.^2\frac{\pi}{2n}\ldots 2\sin.^2\frac{(n-1)\pi}{n}$$

$$= (2^{n-1})^2\,.\,\sin.^2\frac{\pi}{2n}\ldots\sin.^2\frac{(n-1)\pi}{2n}$$

$$\therefore \sqrt{n} = 2^{n-1}\sin.\frac{\pi}{2n}\ldots\sin.\frac{(n-1)\pi}{2n}.$$

Prove the following examples :—

1. $\sqrt{2n+1} = 2^n\sin.\dfrac{\pi}{2n+1}\sin.\dfrac{2\pi}{2n+1}\ldots\sin.\dfrac{n\pi}{2n+1}.$

2. $\sqrt{2} = 2^n\sin.\dfrac{\pi}{4n}\sin.\dfrac{3\pi}{4n}\ldots\sin.\dfrac{(2n-1)\pi}{4n}.$

3. $1 = 2^n\sin.\dfrac{\pi}{2(2n+1)}\sin.\dfrac{3\pi}{2(2n+1)}\ldots\sin.\dfrac{(2n-1)\pi}{2(2n+1)}.$

(7). Solve $y^{2n} - 2y^n\cos.\theta + 1 = 0$, n being any whole number.

$$y^{2n} - 2y^n\cos.\theta + \cos.^2\theta + \sin.^2\theta = 0$$

$$\therefore y^n - \cos.\theta = \pm\sqrt{-1}\sin.\theta;$$

$$\text{or, } y = (\cos.\,\theta \pm \sqrt{-1}\sin.\,\theta)^{\frac{1}{n}}$$

$$= \cos.\frac{\theta}{n} \pm \sqrt{-1}\sin.\frac{\theta}{n}$$

(By Demoivre's Theorem.)

$$\therefore\ y^2 - 2\,y\cos.\frac{2\,r\,\pi + \theta}{n} + 1 = 0$$

an equation which includes all the quadratic factors of the given equation by giving to r the values 0, 1, 2, 3, &c., respectively.

$$\begin{aligned}\therefore\ y^{2n} - 2\,y^n\cos.\,\theta + 1 = {} & \left(y^2 - 2\,y\cos.\frac{\theta}{n} + 1\right) \\ & \times\left(y^2 - 2\,y\cos.\frac{2\pi+\theta}{n} + 1\right) \\ & \times\left(y^2 - 2\,y\cos.\frac{4\pi+\theta}{n} + 1\right) \\ & \vdots \\ & \times\left(y^2 - 2\,y\cos.\frac{(2n-2)\pi+\theta}{n} + 1\right) \quad . \ . \ (a).\end{aligned}$$

By making various suppositions with respect to y and θ various interesting results may be obtained.

EXAMPLES.

1. If $y = 1\ \therefore\ \sin.\frac{\theta}{2} = 2^{n-1}\sin.\frac{\theta}{2\,n}\sin.\frac{2\,\pi+\theta}{2\,n}\ldots\ldots$

$$\sin.\frac{(2\,n-2)\,\pi+\theta}{2\,n}.$$

2. If $y = -1$

$$\therefore 2^{n-1} \cos. \frac{\theta}{2n} \cos. \frac{2\pi+\theta}{2n} \ldots \cos. \frac{(2n-2)\pi+\theta}{2n}$$

$$= \cos. \frac{\theta}{2}, \text{ or, } \sin. \frac{\theta}{2}; \text{ according as } n \text{ is odd or even.}$$

3. $$\text{Tan.} \frac{\theta}{2} = \tan. \frac{\theta}{2n} \tan. \frac{2\pi+\theta}{2n} \ldots \tan. \frac{(2n-2)\pi+\theta}{2n}.$$

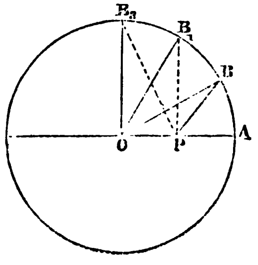
When n is odd,—

4. $$1 = \tan. \frac{\pi}{4n} \tan. \frac{5\pi}{4n} \ldots \tan. \frac{(4n-3)\pi}{4n}.$$

5. $$x^{2n} - 2x^n a^n \cos. \theta + a^{2n} = \left(x^2 - 2xa\cos. \frac{\theta}{n} + a^2\right)$$
$$\times \left(x^2 - 2xa\cos. \frac{2\pi+\theta}{n} + a^2\right)$$
$$\times \left(x^2 - 2xa\cos. \frac{4\pi+\theta}{n} + a^2\right)$$
$$\vdots$$
$$\times \left(x^2 - 2xa\cos. \frac{(2n-2)\pi+\theta}{n} + a^2\right).$$

(8). Demoivre's property of the circle.

Divide the circumference 2π into n equal portions, beginning at B.



Put, $A O B = \theta$

$$\therefore A O B_1 = \theta + \frac{2\pi}{n};\ A O B_2 = \theta + \frac{4\pi}{n}, \text{ \&c.}$$

$$\therefore P B^2 = O P^2 - 2 O P \cos. \theta + 1$$

$$P B_1^2 = O P^2 - 2 O P \cos. \left(\theta + \frac{2\pi}{n}\right) + 1$$

$$P B_2^2 = O P^2 - 2 O P \cos. \left(\theta + \frac{4\pi}{n}\right) + 1$$

&c., &c., &c.

By the property in last article we have,

$O P^{2n} - 2 O P^n \cos. n\theta + 1 = P B^2 . P B_1^2 . P B_2^2 \ldots$ to n factors.

By moving P to various points the following examples may be proved :—

EXAMPLES.

1. Let P be moved to A;

$$\therefore 2 \sin. \frac{n\theta}{2} = A B . A B_1 . A B_2 \ldots \text{ to } n \text{ factors.}$$

2. Let P be moved to P_1 on the line O B by making θ equal to zero;

$$\therefore O P_1^n - 1 = P_1 B . P_1 B_1 . P B_2 \ldots \text{ to } n \text{ factors.}$$

3. Let $n\theta = \pi$;

$$\therefore O P^n + 1 = P B . P B_1 . P B_2 \ldots \text{ to } n \text{ factors.}$$

The properties 2 and 3 are called "Cotes's properties of the circle."

(9). Solve $\sin. x = n \sin. (x + a)$ by series.

By equation (3) :—

$$\epsilon^{x\sqrt{-1}} - \epsilon^{-x\sqrt{-1}} = n\left(\epsilon^{(x+a)\sqrt{-1}} - \epsilon^{-(x+a)\sqrt{-1}}\right).$$

from which we readily derive,

$$\epsilon^{2x\sqrt{-1}}\left(1 - n\,\epsilon^{a\sqrt{-1}}\right) = \left(1 - n\,\epsilon^{-a\sqrt{-1}}\right).$$

Take the log. of both sides of this equation.

$$\therefore 2x\sqrt{-1} = \log.\left(1 - n\,\epsilon^{-a\sqrt{-1}}\right) - \log.\left(1 - n\,\epsilon^{a\sqrt{-1}}\right)$$

$$= n\left\{\epsilon^{a\sqrt{-1}} - \epsilon^{-a\sqrt{-1}}\right\} + \frac{n^2}{2}\left\{\epsilon^{2a\sqrt{-1}} - \epsilon^{-2a\sqrt{-1}}\right\}$$

$$+ \frac{n^3}{3}\left\{\epsilon^{3a\sqrt{-1}} - \epsilon^{-3a\sqrt{-1}}\right\} + \&c.$$

$$\therefore x = n \sin. a + \frac{n^2 \sin. 2a}{2} + \frac{n^3 \sin. 3a}{3} + \&c.$$

EXAMPLES.

1. If $\cos. x = n \cos. (x + a)$

$$\therefore x = \frac{\pi}{2} + n \sin. a + \frac{n^2 \sin. 2a}{2} + \frac{n^3 \sin. 3a}{3} + \&c.$$

2. If $(1 + n) \tan. x = (1 - n) \tan. (x + a)$

$$\therefore a = n \sin. 2x + \frac{n^2}{2} \sin. 4x + \frac{n^3}{3} \sin. 6x + \&c.$$

3. In any triangle, $\sin. C = \frac{c}{a} \sin. (C + B)$;

$$\therefore C = \frac{c}{a} \sin. B + \frac{c^2}{2a^2} \sin. 2 B + \frac{c^3}{3a^3} \sin. 3 B + \ldots.$$

(10). Log. $\epsilon^{x\sqrt{-1}}$

$$= \log. \epsilon^{x\sqrt{-1}} \left(\frac{1 + \epsilon^{x\sqrt{-1}}}{1 + \epsilon^{x\sqrt{-1}}}\right) = \log. \left(\frac{1 + \epsilon^{x\sqrt{-1}}}{1 + \epsilon^{-x\sqrt{-1}}}\right)$$

$$= \log. (1 + \epsilon^{x\sqrt{-1}}) - \log. (1 + \epsilon^{-x\sqrt{-1}})$$

$$= (\epsilon^{x\sqrt{-1}} - \epsilon^{-x\sqrt{-1}}) - \frac{1}{2} (\epsilon^{2x\sqrt{-1}} - \epsilon^{\ 2x\sqrt{-1}})$$

$$+ \frac{1}{3} (\epsilon^{3x\sqrt{-1}} - \epsilon^{-3x\sqrt{-1}}) - \&c.;$$

$$\therefore x = 2 \left(\sin. x - \frac{1}{2} \sin. 2 x + \frac{1}{3} \sin. 3 x - \&c.\right)$$

Again,

$$1 = \frac{1}{1 + \epsilon^{x\sqrt{-1}}} + \frac{1}{1 + \epsilon^{-x\sqrt{-1}}}$$

$$= 1 - \epsilon^{x\sqrt{-1}} + \epsilon^{2x\sqrt{-1}} - \epsilon^{3x\sqrt{-1}} + \&c.$$

$$+ 1 - \epsilon^{-x\sqrt{-1}} + \epsilon^{-2x\sqrt{-1}} - \epsilon^{-3x\sqrt{-1}} + \&c.$$

$$= 2 - (\epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}}) + (\epsilon^{2x\sqrt{-1}} + \epsilon^{-2x\sqrt{-1}}) - \&c.$$

$$\therefore 1 = 2 (\cos. x - \cos. 2 x + \cos. 3 x - \cos. 4 x + \&c.)$$

Again,

$$\frac{1}{1+\epsilon^{x\sqrt{-1}}}-\frac{1}{1+\epsilon^{-x\sqrt{-1}}}=-\frac{(\epsilon^{x\sqrt{-1}}-\epsilon^{-x\sqrt{-1}})}{2+(\epsilon^{x\sqrt{-1}}+\epsilon^{-x\sqrt{-1}})}\ . \ (a)$$

$$\therefore -\frac{(\epsilon^{x\sqrt{-1}}-\epsilon^{-x\sqrt{-1}})}{2+(\epsilon^{x\sqrt{-1}}+\epsilon^{-x\sqrt{-1}})}=1-\epsilon^{x\sqrt{-1}}+\epsilon^{2x\sqrt{-1}}-\epsilon^{3x\sqrt{-1}}+\&c.$$
$$-1+\epsilon^{-x\sqrt{-1}}-\epsilon^{-2x\sqrt{-1}}+\epsilon^{-3x\sqrt{-1}}-\&c.$$

By developing the left-hand side of the equation (a) :—

$$= -(\epsilon^{x\sqrt{-1}}-\epsilon^{-x\sqrt{-1}})+(\epsilon^{2x\sqrt{-1}}-\epsilon^{-2x\sqrt{-1}})-\&c.$$

$$\therefore \tan.\frac{x}{2}=2(\sin. x-\sin. 2x+\sin. 3x-\sin. 4x+\&c.)$$

The following examples may be readily proved by putting $\frac{\pi}{2}-x$ for x in the above formulæ.

1. $\pi-2x=4\left(\cos. x-\frac{1}{2}\sin. 2x-\frac{1}{3}\cos. 3x+\&c.\right)$

2. $1=2(\sin. x+\cos. 2x-\sin. 3x-\cos. 4x+\&c.)$

3. $\text{Sec.} x-\tan. x=2(\cos. x-\sin. 2x-\cos. 3x+\sin. 4x-\&c.)$

Put, $\pi-x$ for x, then,

4. $\pi-x=2\left(\sin. x+\frac{1}{2}\sin. 2x+\frac{1}{3}\sin. 3x+\frac{1}{4}\sin. 4x+\&c.\right)$

5. $-1=2(\cos. x+\cos. 2x+\cos. 3x+\cos. 4x+\&c.)$

6. $\text{Cot.}\frac{x}{2}=2(\sin. x+\sin. 2x+\sin. 3x+\sin. 4x+\&c.)$

(11). In any triangle $a^2 = b^2 + c^2 - 2\,b\,c\cos. A$.

$$\text{Or, } a^2 = b^2 + c^2 - b\,c\,(\epsilon^{A\sqrt{-1}} + \epsilon^{-A\sqrt{-1}})$$

$$= b^2\left\{1 + \frac{c^2}{b^2} - \frac{c}{b}\epsilon^{A\sqrt{-1}} - \frac{c}{b}\epsilon^{-A\sqrt{-1}}\right\}$$

$$= b^2\left(1 - \frac{c}{b}\epsilon^{A\sqrt{-1}}\right)\left(1 - \frac{c}{b}\epsilon^{-A\sqrt{-1}}\right)$$

$$\therefore 2\log.\frac{a}{b} = \log.\left(1 - \frac{c}{b}\epsilon^{A\sqrt{-1}}\right) + \log.\left(1 - \frac{c}{b}\epsilon^{-A\sqrt{-1}}\right)$$

$$\text{or, } \log.\frac{b}{a} = \frac{c}{b}\cos. A + \frac{c^2}{2\,b^2}\cos. 2\,A + \frac{c^3}{3\,b^3}\cos. 3\,A + \&c.$$

EXAMPLES.

1. When $c = b$, or the triangle is isosceles;

$$\therefore \log.\left(\frac{1}{2}\operatorname{cosec.}\frac{A}{2}\right) = \cos. A + \frac{1}{2}\cos. 2A + \frac{1}{3}\cos. 3A + \&c.$$

2. Take (in example 1) $\pi - A$ for A;

$$\therefore \log.\left(2\cos.\frac{A}{2}\right) = \cos. A - \frac{1}{2}\cos. 2\,A + \frac{1}{3}\cos. 3\,A - \&c.$$

(Todhunter's Trig., p. 247.)

(12). In any triangle $\dfrac{a}{c} = \dfrac{\sin. A}{\sin. C}$;

$$\therefore \frac{a}{c} = \frac{\epsilon^{A\sqrt{-1}} - \epsilon^{-A\sqrt{-1}}}{\epsilon^{C\sqrt{-1}} - \epsilon^{-C\sqrt{-1}}} = \frac{\epsilon^{-A\sqrt{-1}} - \epsilon^{A\sqrt{-1}}}{\epsilon^{-C\sqrt{-1}} - \epsilon^{C\sqrt{-1}}}.$$

$$\text{or, } \frac{a^2}{c^2} = \frac{(\epsilon^{A\sqrt{-1}} - \epsilon^{-A\sqrt{-1}})\,(\epsilon^{-A\sqrt{-1}} - \epsilon^{A\sqrt{-1}})}{(\epsilon^{C\sqrt{-1}} - \epsilon^{-C\sqrt{-1}})\,(\epsilon^{-C\sqrt{-1}} - \epsilon^{C\sqrt{-1}})}$$

$$\therefore 2 \log. \frac{a}{c} = \log.(\epsilon^{A\sqrt{-1}} - \epsilon^{-A\sqrt{-1}}) + \log.(\epsilon^{-A\sqrt{-1}} - \epsilon^{A\sqrt{-1}})$$
$$- \log.(\epsilon^{C\sqrt{-1}} - \epsilon^{-C\sqrt{-1}}) - \log.(\epsilon^{-C\sqrt{-1}} - \epsilon^{C\sqrt{-1}});$$

$$\text{or, } \log. \frac{a}{c} = (\cos. 2\,C - \cos. 2\,A) + \frac{1}{2}(\cos. 4\,C - \cos. 4\,A)$$
$$+ \frac{1}{3}(\cos. 6\,C - \cos. 6\,A) + \&c.$$

EXAMPLES.

1. $$\text{Log.} \frac{a}{c} = 2 \left\{ \sin.^2 A - \sin.^2 C + \frac{1}{2}(\sin.^2 2\,A - \sin.^2 2\,C) + \frac{1}{3}(\sin.^2 3\,A - \sin.^2 3\,C) + \&c. \right\}.$$

2. When $A = B$, then,

$$\log. 2 \cos. A = 2 \left\{ \sin. 3\,A \sin. A + \frac{1}{2} \sin. 6\,A \sin. 2\,A + \frac{1}{3} \sin. 9\,A \sin. 3\,A + \&c. \right\}.$$

3. When $A = \frac{\pi}{2}$, then,

$$\log.(\text{cosec.} C) = 2 \left\{ \cos.^2 C - \frac{1}{2} \sin.^2 2\,C + \frac{1}{3} \cos.^2 3\,C - \&c. \right\}.$$

(13). Let $S = \sin.(a + 2x) + \sin.(a + 4x) + \ldots \sin.(a + 2nx)$ where (a) and (x) are any angles and (n) the number of terms taken.

Since, $\cos.(a + x) - \cos.(a + 3x) = 2\sin.(a + 2x)\sin.x$

and, $\cos.(a + 3x) - \cos.(a + 5x) = 2\sin.(a + 4x)\sin.x$

" $\cos.(a + 5x) - \cos.(a + 7x) = 2\sin.(a + 6x)\sin.x$

$\vdots \qquad \vdots \qquad \vdots$

$\cos.\{a + (2n-1)x\} - \cos.\{a + (2n+1)x\} = 2\sin.(a + 2nx)\sin.x.$

Add these equations together, then the first and last terms on the left-hand side remain. Therefore,

$\text{Cos.}(a + x) - \cos.\{a + (2n + 1)x\} = 2S\sin.x;$

$$\text{or, } S = \frac{\cos.(a + x) - \cos.\{a + (2n + 1)x\}}{2\sin.x}$$

$$= \frac{\sin.\{a + (n + 1)x\}\sin.nx}{\sin.x}.$$

Examples which follow by taking various values for a and x:—

1. $\text{Sin.}x + \sin.2x + \sin.3x + \text{to } n \text{ terms} = \dfrac{\sin.\frac{n+1}{2}x\sin.\frac{nx}{2}}{\sin.\frac{x}{2}}.$

2. $\text{Sin.}x + \sin.3x + \sin.5x + \text{to } n \text{ terms} = \dfrac{\sin.^2 nx}{\sin.x}.$

3. $\text{Cos.}x + \text{cos.}2x + \text{cos.}3x +$ to n terms $= \dfrac{\text{cos.}\frac{n+1}{2}x \,\text{sin.}\frac{n x}{2}}{\text{sin.}\frac{x}{2}}$.

4. $\text{Cos.}\,(a + 2x) + \text{cos.}\,(a + 4x) + \ldots$, to n terms

$$= \frac{\text{cos.}\,\{a + (n+1)\,x\}\ \text{sin.}\,n\,x}{\text{sin.}\,x}.$$

5. $\text{Cos.}\,x + \text{cos.}\,3\,x + \text{cos.}\,5\,x +$ to n terms $= \dfrac{\text{sin.}\,2\,n\,x}{2\ \text{sin.}\,x}$.

6. From 2 and 5.

$$\text{Tan.}\,n\,x = \frac{\text{sin.}\,x + \text{sin.}\,3\,x + \text{sin.}\,5\,x + \text{to } n \text{ terms}}{\text{cos.}\,x + \text{cos.}\,3\,x + \text{cos.}\,5\,x + \text{to } n \text{ terms}}.$$

(14). Let $S = \text{cosec.}x + \text{cosec.}2x + \text{cosec.}4x + \ldots$ to n terms:—

$$\text{Since, cot.}\,\frac{x}{2} - \text{cot.}\,x = \text{cosec.}\,x$$

$$\therefore \text{cot.}\,x - \text{cot.}\,2\,x = \text{cosec.}\,2\,x\,;$$

$$\text{and, cot.}\,2\,x - \text{cot.}\,4\,x = \text{cosec.}\,4\,x\,;$$

$$\text{and, cot.}\,4\,x - \text{cot.}\,8\,x = \text{cosec.}\,8\,x\,;$$

$$\vdots \qquad\qquad \vdots$$

$$\text{cot.}\,2^{n-2}\,x - \text{cot.}\,2^{n-1}\,x = \text{cosec.}\,(2^{n-1}\,x)\,;$$

$$\therefore S = \text{cot.}\,\frac{x}{2} - \text{cot.}\,2^{n-1}\,x.$$

EXAMPLES.

1. From, $\tan. x = \cot. x - 2 \cot. 2 x$;

$$s = \tan. x + 2 \tan. 2 x + \dots. 2^{n-1} \tan. 2^{n-1} x$$
$$\text{where } s = \cot. x - 2^n \cot. 2^n x$$

$$s^1 = \frac{1}{2} \tan. \frac{x}{2} + \frac{1}{4} \tan. \frac{x}{4} + \dots. \frac{1}{2^n} \tan. \frac{x}{2^n}$$

$$\text{where } s^1 = \frac{1}{2^n} \cot. \frac{x}{2^n} - \cot. x.$$

When n is infinite :—

$$\therefore \frac{1}{2} \tan. \frac{x}{2} + \frac{1}{4} \tan. \frac{x}{4} + \dots \text{ to infinity} = \frac{1}{x} - \cot. x.$$

2. From, $\tan. (n+1) x - \tan. n x = \sin. x \sec. n x \sec. (n+1) x$;

$$\therefore \sec. x \sec. 2 x + \sec. 2 x \sec. 3 x + \dots \sec. n x \sec. (n+1) x$$
$$= \frac{\tan. (n+1) x - \tan x}{\sin. x}.$$

(15). Let $S = \cos. x + \frac{a^2 \cos. 2 x}{1.2} + \frac{a^3 \cos. 3 x}{1.3.3} + \left\{ \begin{array}{l} \text{\&c., to} \\ \text{infinity.} \end{array} \right.$

Since, $2 \cos. x = \epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}}$;

$$\therefore \frac{2 a^2 \cos. 2 x}{1.2} = \frac{(a \epsilon^{x\sqrt{-1}})^2}{1.2} + \frac{(a \epsilon^{-x\sqrt{-1}})^2}{1.2}.$$

$$\frac{2 a^3 \cos. 3 x}{1.2.3} = \frac{(x \epsilon^{x\sqrt{-1}})^3}{1.2.3} + \frac{(a \epsilon^{-x\sqrt{-1}})^3}{1.2.3}.$$

&c. &c. &c.

Add these equations together, and observe that—

$$\epsilon^y = 1 + y + \frac{y^2}{1 . 2} + \frac{y^3}{1 . 2 . 3} + \&c.$$

$$\therefore 2 S + 2 = \epsilon^{(a \epsilon^{x\sqrt{-1}})} + \epsilon^{(a \epsilon^{-x\sqrt{-1}})}.$$

$$= \epsilon^{a (\cos. x + \sqrt{-1} \sin. x)} + \epsilon^{a (\cos. x - \sqrt{-1} \sin. x)}$$

$$= \epsilon^{a \cos. x} . \epsilon^{a \sin. x \sqrt{-1}} + \epsilon^{a \cos.} \epsilon^{-a \sin. x \sqrt{-1}}$$

$$= \epsilon^{a \cos. x} \left(\epsilon^{a \sin. x \sqrt{-1}} + \epsilon^{-a \sin. x \sqrt{-1}}\right)$$

$$\therefore S + 1 = \epsilon^{a \cos. x} \cos. (a \sin. x).$$

EXAMPLES.

1. $a \sin. x + \frac{a^2 \sin. 2 x}{1 . 2} + \ldots$ to infinity $= \epsilon^{a \cos. x} \sin. (a \sin. x)$.

2. $a \sin. x - \frac{a^2 \sin. 2 x}{2} + \frac{a^3 \sin. 3 x}{3} -$ to infinity

$$= \tan.^{-1} \frac{a \sin. x}{1 + a \cos. x}.$$

3. $a \sin. x + a^2 \sin. 2 x +$ to infinity $= \frac{a \sin x}{1 - 2 a \cos. x + a^2}$.

4. $a \cos. x + a^2 \cos. 2 x +$ to infinity

$$= \frac{1}{2} \left(\frac{1 - a^2}{1 - 2 a \cos. x + a^2} - 1\right).$$

The examples 3 and 4 may be obtained more simply perhaps by the division of—

$\frac{1}{1 - a \epsilon^{x\sqrt{-1}}}$ and $\frac{1}{1 - a \epsilon^{-x\sqrt{-1}}}$, and then adding, &c.

USEFUL DEVELOPMENTS.

1. $\text{Sin.}\, x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.$

2. $\text{Cos.}\, x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.$

3. $\text{Log.}\, (a + x) = \log.\, a + \frac{x}{a} - \frac{x^2}{2\, a^2} + \frac{x^3}{3\, a^5} - \&c.$

4. $a^x = 1 + x \log.\, a + \frac{x^2 \log.\, ^2 a}{1.2} + \frac{x^3 \log.\, ^3 a}{1.2.3} + \&c.$

5. $\text{Tan.}\, x = x + \frac{x^3}{1.3} + \frac{2\, x^5}{3.5} + \frac{17\, x^7}{3.5.7.9}.$

6. $\text{Sin.}^{-1}\, x = x + \frac{x^3}{1.2.3} + \frac{1.3\, x^5}{2.4.5} + \frac{1.3.5\, x^7}{2.4.6.7} + \&c.$

7. $\text{Cos.}^{-1}\, x = \frac{\pi}{2} - x - \frac{x^3}{2.3} - \frac{1.3\, x^5}{2.4.5} - \frac{1.3.5\, x^7}{2.4.6.7} - \&c.$

8. $\text{Tan.}^{-1}\, x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

9. $\epsilon = 2.7182818$ $\sqrt{2} = 1.4142135.$

10. $\sqrt{3} = 1.7320508$ $\sqrt{5} = 2.2360679.$